Hybrid Monte Carlo methods in computational finance

Dissertation defense

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Monte Carlo methods in computational finance

- Monte Carlo methods are highly appreciated and intensively employed in computational finance.
- Applications like financial derivatives valuation or risk management.
- Advantages:
  - Easy interpretation.
  - Flexibility.
  - Straightforward implementation.
  - Easily extended to multi-dimensional problems.
- The latter feature of Monte Carlo methods is a clear advantage over other competing numerical methods. For problems of more than five dimensions, Monte Carlo method is the only possible choice.
- The main drawback is the rather poor balance between computational cost and accuracy.

Financial derivatives valuation

- One of the main areas in quantitative finance.
- A derivative is a contract between two or more parties based on one asset or more assets. Its value is determined by fluctuations in the underlying asset.
- Mathematically, the underlying financial assets are modelled by means of *stochastic processes* and *Stochastic Differential Equations* (SDEs).
- The derivative price can be represented by the solution of a partial differential equation (PDE) via *Itô's lemma*.
- Due to the celebrated *Feynman-Kac theorem*, the solution of many PDEs appearing in derivative valuation can be written in terms of a probabilistic representation.
- The associated risk neutral asset price density function plays an important role.

Feynman-Kac theorem

Theorem (Feynman-Kac)

Let $V(t, S)$ be a sufficiently differentiable function of time t and stock price $S(t)$. Suppose that $V(t, S)$ satisfies the following PDE, with drift term, $\mu(t, S)$, volatility term, $\sigma(t, S)$, and r the risk-free rate:

$$\frac{\partial V}{\partial t} + \mu(t, S)\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2(t, S)\frac{\partial^2 V}{\partial S^2} - rV = 0,$$

with final condition $h(T, S)$. The solution for $V(t, S)$ at time $t_0 < T$ is then

$$V(t_0, S) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t_0)}h(T, S)|\mathcal{F}(t_0)\right],$$

where the expectation is taken under measure $\mathbb{Q}$, with respect to the process:

$$\mathrm{d}S(t) = \mu^{\mathbb{Q}}(t, S)\mathrm{d}t + \sigma(t, S)\mathrm{d}W^{\mathbb{Q}}(t), \quad \text{for} \quad t > t_0.$$

- The expectation, written in integral form, results in the *risk-neutral valuation formula*,

$$V(t_0, S) = \mathrm{e}^{-r(T-t_0)}\int_{\mathbb{R}} h(T, y)f(y|\mathcal{F}(t_0))\mathrm{d}y,$$

with $f(\cdot)$ the density of the underlying process.

Monte Carlo method

- Monte Carlo methods are numerical techniques to evaluate integrals, based on the analogy between probability and volume.
- Suppose we need to compute an integral

$$I := \int_C g(x)\mathrm{d}x,$$

- Independent and identically distributed samples in C, $X_1, X_2, \ldots, X_n$.
- The definition a Monte Carlo estimator is

$$\bar{I}_n := \frac{1}{n}\sum_{j=1}^{n} g(X_j).$$

Monte Carlo method

- If g is integrable over C, by the *strong law of the large numbers*,

$$\bar{I}_n \to I \quad \text{as} \quad n \to \infty,$$

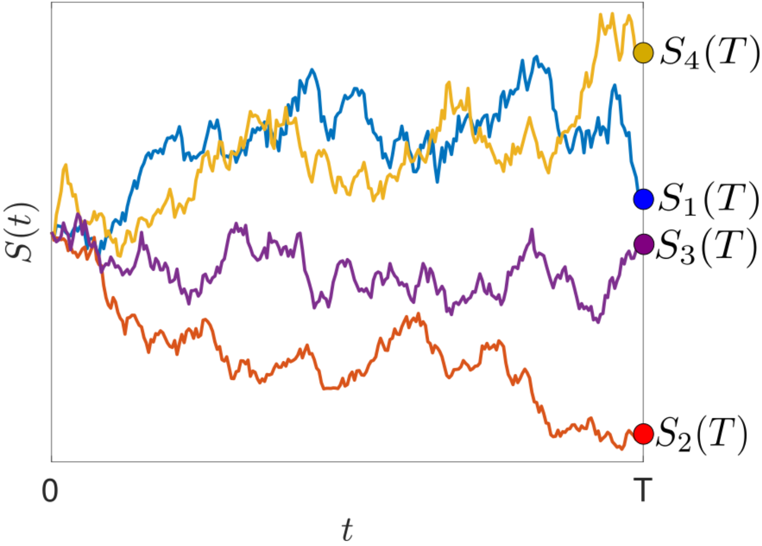
  with probability one.
- Furthermore, if g is square integrable, we can define

$$s_g := \sqrt{\int_C (g(x) - I)^2 \, \mathrm{d}x}$$

- By the *central limit theorem*, the error of the Monte Carlo estimate $I - \bar{I}$ is assumed normally distributed with mean 0 and standard deviation $s_g/\sqrt{n}$.
- Therefore, the order of convergence of the plain Monte Carlo method is $\mathcal{O}(1/\sqrt{n})$.

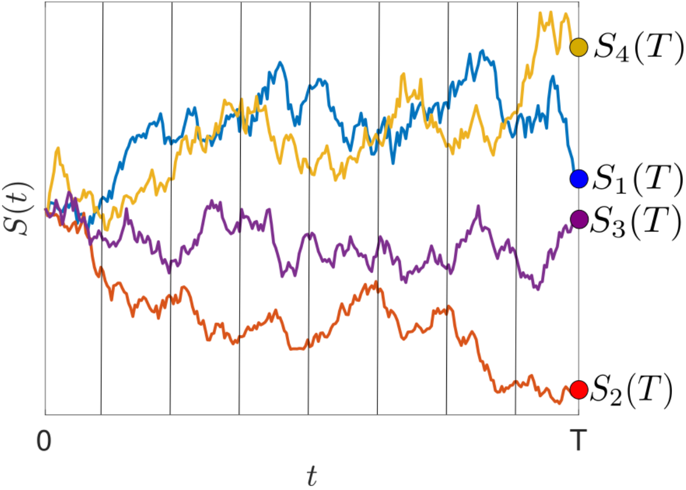
Monte Carlo method

$$\mathbb{E}[h(T,S)] \approx \frac{1}{4}\sum_i h(S_i(T))$$



Monte Carlo method: What if...?

- The required samples cannot be generated exactly?



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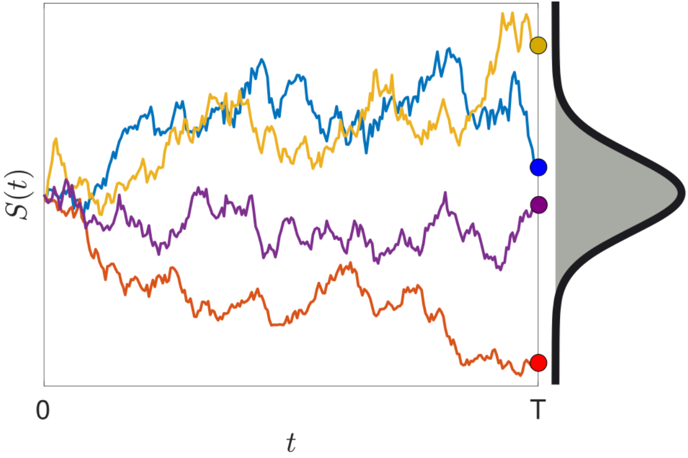

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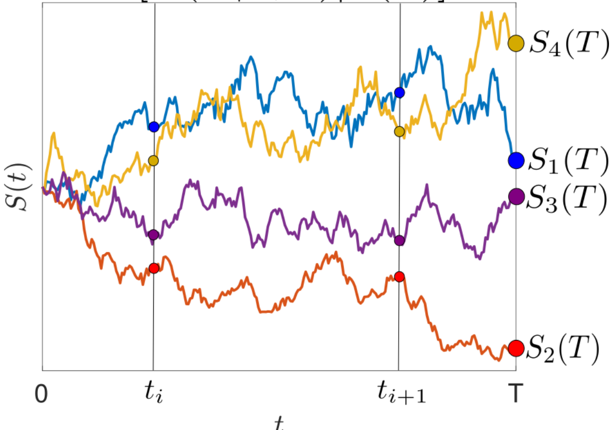
$$\mathbb{E}[V(t_{i+1}, S)|S(t_i)]$$



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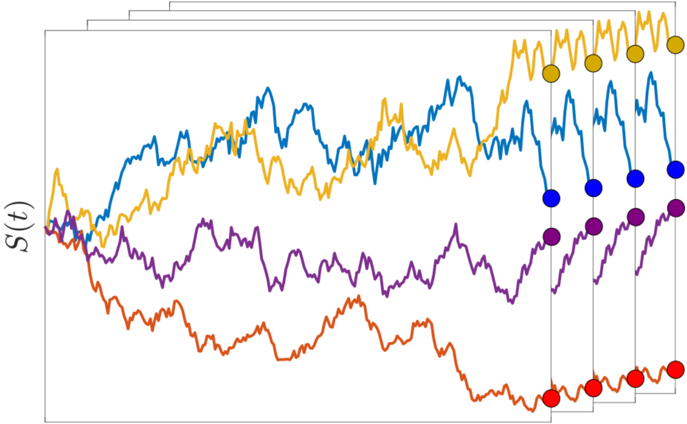

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- You want to handle this issues efficiently, i.e. achieving a good balance between accuracy and computational cost?
- ANSWER: Combine Monte Carlo methods with other mathematical or computational techniques $\rightarrow$ Hybrid solutions.
- And you can do a PhD in The Netherlands, apply hybrid solutions to particular situations in finance while meeting many interesting people.

“Exact” Monte Carlo simulation of the SABR model

- The formal definition of the SABR model reads

$$\begin{aligned}\mathrm{d}S(t) &= \sigma(t)S^{\beta}(t)\mathrm{d}W_S(t), \quad & S(0) &= S_0 \exp\left(rT\right), \\ \mathrm{d}\sigma(t) &= \alpha\sigma(t)\mathrm{d}W_\sigma(t), & \sigma(0) &= \sigma_0.\end{aligned}$$

- $S(t) = \bar{S}(t)\exp\left(r(T-t)\right)$ is the forward price of the underlying $\bar{S}(t)$, with r an interest rate, S_0 the spot price and T the maturity.
- $\sigma(t)$ is the stochastic volatility.
- $W_f(t)$ and $W_\sigma(t)$ are two correlated Brownian motions.
- SABR parameters:
  - The volatility of the volatility, $\alpha > 0$.
  - The CEV elasticity, $0 \leq \beta \leq 1$.
  - The correlation coefficient, ρ ($W_f W_\sigma = \rho t$).

"Exact" Monte Carlo simulation of the SABR model

- Simulation of the volatility, $\sigma(t)|\sigma(s)$:

$$\sigma(t) \sim \sigma(s) \exp\left(\alpha \hat{W}_\sigma(t) - \frac{1}{2}\alpha^2(t-s)\right).$$

- Simulation of the integrated variance, $\int_s^t \sigma^2(z)\mathrm{d}z|\sigma(t),\sigma(s)$.
- Simulation of the asset, $S(t)|S(s), \int_s^t \sigma^2(z)\mathrm{d}z, \sigma(t), \sigma(s)$.
- The conditional integrated variance is a challenging part.
- We propose a one[1] and multiple[2] time-step simulation:
    - Approximate the conditional distribution by using Fourier techniques and copulas.
    - Marginal distribution based on a Fourier inversion method.
    - Conditional distribution based on copulas.
    - Improvements in performance and efficiency.

[1] A. Leitao, L. A. Grzelak, and C. W. Oosterlee (2017a). "On a one time-step Monte Carlo simulation approach of the SABR model: application to European options". In: *Applied Mathematics and Computation* 293, pp. 461–479. ISSN: 0096-3003. DOI: http://dx.doi.org/10.1016/j.amc.2016.08.030

[2] A. Leitao, L. A. Grzelak, and C. W. Oosterlee (2017b). "On an efficient multiple time step Monte Carlo simulation of the SABR model". In: *Quantitative Finance*. DOI: 10.1080/14697688.2017.1301676. eprint: http://dx.doi.org/10.1080/14697688.2017.1301676

The data-driven COS method

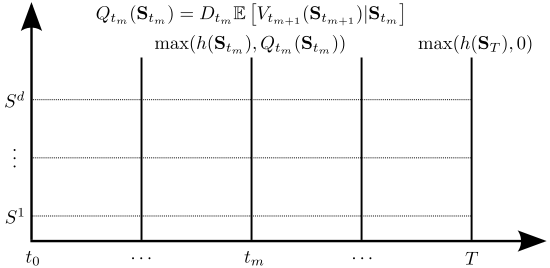
- We aim to recover closed-form expressions for the density and distribution functions given the samples.
- We extend the well-known COS method to the cases when the characteristic function is not available as, for example, the SABR model.
- We base our approach in the density estimation problem in the framework of the so-called *Statistical learning theory*.
- We exploit the connection between orthogonal series estimators and the COS method.
- Chapter 4 of the thesis and submitted for publication.

Applications of the data-driven COS method

- Particularly useful for risk management, where not only the expectation but also the extreme cases need to be consider.
- As the method is based on data (samples), it is more generally applicable.
- Some applications:
  - Efficient computation of the *sensitivities* of a financial derivative, commonly known as *Greeks*.
  - Efficient computation of risk measures like *Value-at-Risk* (VaR) and *Expected Shortfall* (ES).

GPU acceleration of SGBM

- SGBM is a method to price multi-dimensional early-exercise derivatives.
- Early-exercise basket options:

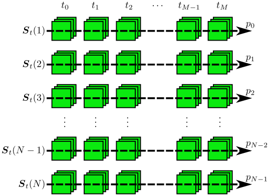


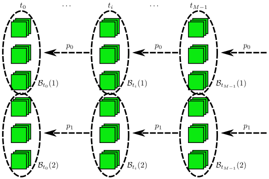
- We aim to increase the problem dimension drastically.
- We propose[3] the parallelization of the method.
- General-Purpose computing on Graphics Processing Units (GPGPU).

[3] A. Leitao and C. W. Oosterlee (2015). "GPU Acceleration of the Stochastic Grid Bundling Method for Early-Exercise options". In: *International Journal of Computer Mathematics* 92.12, pp. 2433–2454. DOI: 10.1080/00207160.2015.1067689

GPU acceleration of SGBM

- Parallel strategy: two parallelization stages:
  - Forward: Monte Carlo simulation.
  - Backward: Bundles → Opportunity of parallelization.
- Novelty in early-exercise option pricing methods.





- Very high-dimensional problems (up to 50D) can be efficiently and accurately treated.
- Our GPU parallel version of the SGBM is 100 times faster than the sequential version.

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- My family.