SABR model

Álvaro Leitao

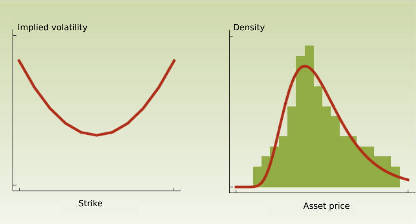
Lecture group, CWI

November 18, 2013

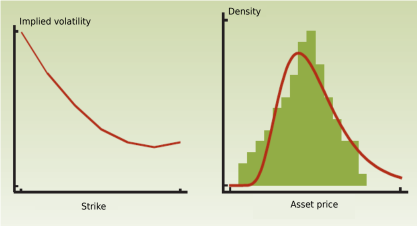
Outline

Introduction

- Since 70's: Black-Scholes.
  - Standar option pricing method. Hypothesis:
    - ★ The price follows lognormal distribution.
    - ★ The volatility is constant.
  - Crisis 1987. Model problems.



(a) Smile

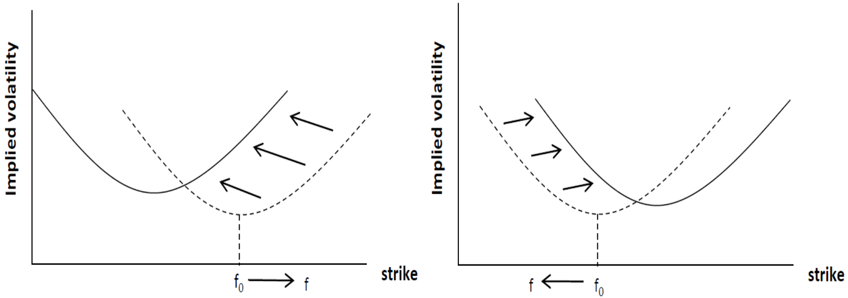


(b) Skew

- Models which modify the price distribution.
- Models which allow non-constant volatility.
  - Local volatility models: Dupire.
  - Stochastic volatility models: Heston or SABR.

Local vs. Stochastic volatility models

- LVM volatility is a function.
- Both capture smile well.
- Both can be used for pricing.
- LVM show an opposite dynamic.



- LVM problems with risk measures.
- SVM solve it. Volatility also follows a stochastic process.

SABR model

SABR model (Hagan et al. 2002)

$$dF_t = \alpha_t F_t^\beta dW_t^1, \qquad F_0 = \hat{f}$$
$$d\alpha_t = \nu \alpha_t dW_t^2, \qquad \alpha_0 = \alpha$$

- Forward, $F_t = S_t e^{(r-q)(T-t)}$, where r is constant interest rate, q constant dividend yield and T maturity date.
- Volatility, α_t.
- dW_t^1 y dW_t^2, correlated geometric brownian motions:

$$dW_1 dW2 = \rho dt$$

- Inicial values: S_0 y α.
- Model parameters: α, β, ν and ρ.
- S-tochastic A-lpha B-eta R-ho model.

SABR model - Implied volatility

$$\sigma_B(K,\hat{f},T) = \frac{\alpha}{(K\hat{f})^{(1-\beta)/2}\left[1+\frac{(1-\beta)^2}{24}\ln^2\left(\frac{\hat{f}}{K}\right)+\frac{(1-\beta)^4}{1920}\ln^4\left(\frac{\hat{f}}{K}\right)+\cdots\right]}\cdot\left(\frac{z}{x(z)}\right)\cdot$$

$$\left[1+\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(K\hat{f})^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(K\hat{f})^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right]\cdot T+\cdots.$$

Note that the previous expression depends on the parameters K, $\hat{f}$ and T, also through the functions:

$$z = \frac{\nu}{\alpha}(K\hat{f})^{(1-\beta)/2}\ln\left(\frac{\hat{f}}{K}\right),$$

and

$$x(z) = \ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right).$$

SABR model - Obloj correction (2008)

$$\sigma_B(K,\hat{f},T) = \frac{1}{\left[1+\frac{(1-\beta)^2}{24}\ln^2\left(\frac{\hat{f}}{K}\right)+\frac{(1-\beta)^4}{1920}\ln^4\left(\frac{\hat{f}}{K}\right)+\cdots\right]}\cdot\left(\frac{\nu\ln\left(\frac{\hat{f}}{K}\right)}{x(z)}\right)\cdot$$

$$\left[1+\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(K\hat{f})^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(K\hat{f})^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right]\cdot T+\cdots,$$

where the following new expression for z is considered:

$$z = \frac{\nu\left(\hat{f}^{1-\beta}-K^{1-\beta}\right)}{\alpha(1-\beta)},$$

and $x(z)$ is given by the same previous expression.

- The omitted terms can be neglected.

SABR model - Approx. implied volatility

$$\sigma_B(K, \hat{f}, T) = \frac{1}{\omega}\left(1 + A_1 \ln\left(\frac{K}{\hat{f}}\right) + A_2 \ln^2\left(\frac{K}{\hat{f}}\right) + BT\right),$$

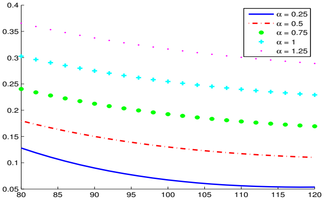
where the coefficients A_1, A_2 and B are given by

$$\begin{aligned} A_1 &= -\frac{1}{2}(1-\beta-\rho\nu\omega), \\ A_2 &= \frac{1}{12}\Big((1-\beta)^2 + 3\big((1-\beta)-\rho\nu\omega\big) + \left(2-3\rho^2\right)\nu^2\omega^2\Big), \\ B &= \frac{(1-\beta)^2}{24}\frac{1}{\omega^2} + \frac{\beta\rho\nu}{4}\frac{1}{\omega} + \frac{2-3\rho^2}{24}\nu^2, \end{aligned}$$

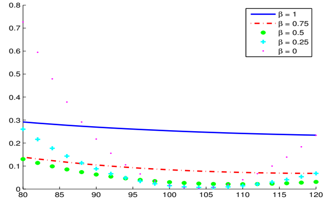
and the value of ω is given by

$$\omega = \frac{\hat{f}^{1-\beta}}{\alpha}.$$

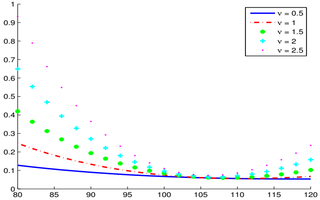
SABR model - Parameters



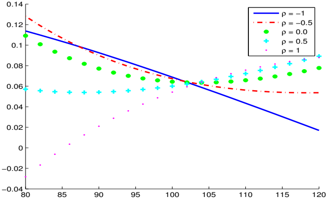
(e) $\alpha > 0$, the volatility's reference level.



(f) $0 \leq \beta \leq 1$, the variance elasticity.



(g) $\nu > 0$, the volatility of the volatility.



(h) $-1 \leq \rho \leq 1$, the correlation coefficient.

▸ SABR parameters

SABR model - Calibration

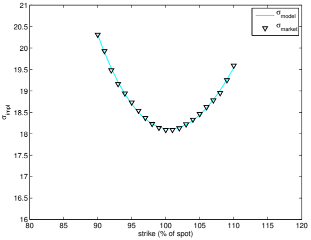
The calibration process tries to obtain a set of model parameters that makes model values as close as possible to market ones, i.e

$$V_{market}(K_j, \hat{f}, T_i) \approx V_{sabr}(K_j, \hat{f}, T_i)$$

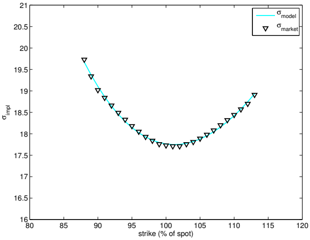
In order to achieve this target we must follow several steps:

- Prices or volatilities.
- Representative market data.
- Error measure.
- Cost function.
- Optimization algorithm.
- Fix parameters on beforehand.
- Calibrate and compare the obtained results.

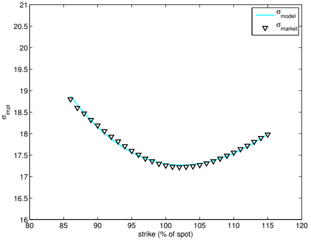
SABR model - Calibration example



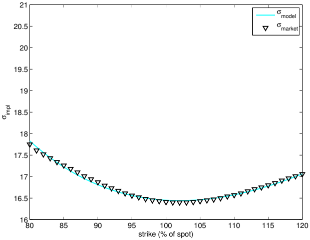
(i) 3 months maturity.



(j) 6 months maturity.



(k) 12 months maturity.



(l) 24 months maturity.

SABR model - Drawback

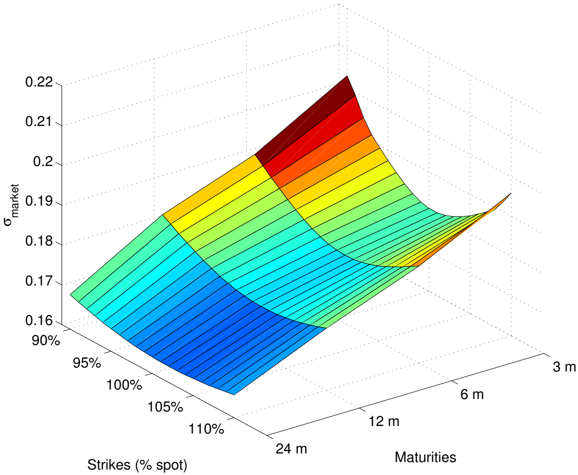
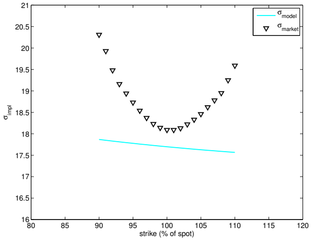
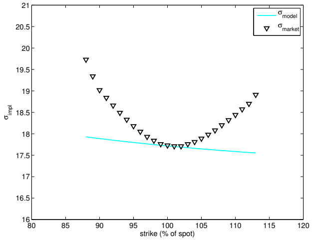


Figure: Market volatility surface.

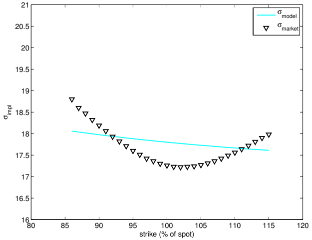
SABR model - Drawback



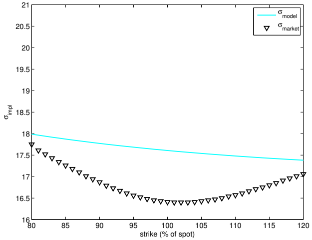
(a) 3 months maturity.



(b) 6 months maturity.



(c) 12 months maturity.



(d) 24 months maturity.

Dynamic SABR model

dynamic SABR model

$$dF_t = \alpha_t F_t^{\beta} dW_t^1, \qquad F_0 = \hat{f}$$
$$d\alpha_t = \nu(t)\alpha_t dW_t^2, \qquad \alpha_0 = \alpha$$

- Forward, F_t.
- Volatility, α_t.
- dW_t^1 y dW_t^2, correlated geometric brownian motions:

$$dW_1 dW2 = \rho(t)dt$$

- Inicial values: S_0 y α.
- Model parameters: α, β and ones that $\nu(t)$ and $\rho(t)$ can provide.
- Approximation of implied volatility provided by Osajima (2007).

Dynamic SABR model

dynamic SABR model

$$dF_t = \alpha_t F_t^{\beta} dW_t^1, \qquad F_0 = \hat{f}$$
$$d\alpha_t = \nu(t)\alpha_t dW_t^2, \qquad \alpha_0 = \alpha$$

- Forward, F_t.
- Volatility, α_t.
- dW_t^1 y dW_t^2, correlated geometric brownian motions:

$$dW_1 dW2 = \rho(t)dt$$

- Inicial values: S_0 y α.
- Model parameters: α, β and ones that $\nu(t)$ and $\rho(t)$ can provide.
- Approximation of implied volatility provided by Osajima (2007).

Dynamic SABR model - Approx. implied volatility

$$\sigma_B(K,\hat{f},T) = \frac{1}{\omega}\left(1 + A_1(T)\ln\left(\frac{K}{\hat{f}}\right) + A_2(T)\ln^2\left(\frac{K}{\hat{f}}\right) + B(T)T\right),$$

where

$$\begin{aligned}
A_1(T) &= \frac{\beta-1}{2} + \frac{\eta_1(T)\omega}{2},\\
A_2(T) &= \frac{(1-\beta)^2}{12} + \frac{1-\beta-\eta_1(T)\omega}{4} + \frac{4\nu_1^2(T) + 3\left(\eta_2^2(T) - 3\eta_1^2(T)\right)}{24}\omega^2,\\
B(T) &= \frac{1}{\omega^2}\left(\frac{(1-\beta)^2}{24} + \frac{\omega\beta\eta_1(T)}{4} + \frac{2\nu_2^2(T) - 3\eta_2^2(T)}{24}\omega^2\right),
\end{aligned}$$

with

$$\nu_1^2(T) = \frac{3}{T^3}\int_0^T (T-t)^2\nu^2(t)dt, \qquad \nu_2^2(T) = \frac{6}{T^3}\int_0^T (T-t)t\nu^2(t)dt,$$

$$\eta_1(T) = \frac{2}{T^2}\int_0^T (T-t)\nu(t)\rho(t)dt, \quad \eta_2^2(T) = \frac{12}{T^4}\int_0^T\int_0^t\left(\int_0^s \nu(u)\rho(u)du\right)^2 dsdt.$$

Dynamic SABR model - $\rho(t)$ and $\nu(t)$ functions

- $\rho(t)$ and $\nu(t)$ have to be smaller for long terms (t large) rather than for short terms (t small).

Constant

- $\rho(t) = \rho_0$
- $\nu(t) = \nu_0$
- α, β, ρ_0, ν_0, SABR model.

Piecewise

- $\rho(t) = \rho_0, t \leq T_0 \quad \rho(t) = \rho_1, t > T_0$
- $\nu(t) = \nu_0, t \leq T_0 \quad \nu(t) = \nu_1, t > T_0$
- α, β, ρ_0, ν_0, ρ_1, ν_1 and T_0

Classical

- $\rho(t) = \rho_0 e^{-at}$
- $\nu(t) = \nu_0 e^{-bt}$
- α, β, ρ_0, ν_0, a and b

General

- $\rho(t) = (\rho_0 + q_\rho t)e^{-at} + d_\rho$
- $\nu(t) = (\nu_0 + q_\nu t)e^{-bt} + d_\nu$
- α, β, ρ_0, ν_0, a, b, d_ρ, d_ν, q_ρ and q_ν.

Dynamic SABR model - Classical choice

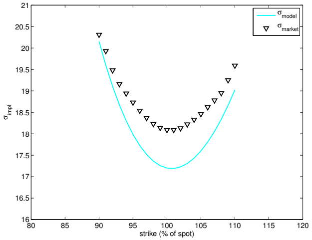
$$\nu_1^2(T) = \frac{6\nu_0^2}{(2bT)^3}\left[\left((2bT)^2/2 - 2bT + 1\right) - e^{-2bT}\right],$$

$$\nu_2^2(T) = \frac{6\nu_0^2}{(2bT)^3}\left[2(e^{-2bT} - 1) + 2bT(e^{-2bT} + 1)\right],$$

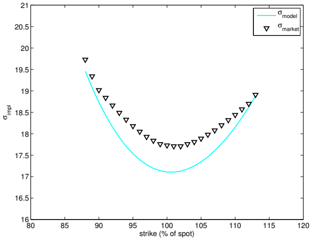
$$\eta_1(T) = \frac{2\nu_0\rho_0}{T^2(a+b)^2}\left[e^{-(a+b)T} - \left(1 - (a+b)T\right)\right],$$

$$\eta_2^2(T) = \frac{3\nu_0^2\rho_0^2}{T^4(a+b)^4}\left[1 - 8e^{-(a+b)T} + \left(7 + 2(a+b)T\left(-3 + (a+b)T\right)\right)\right].$$

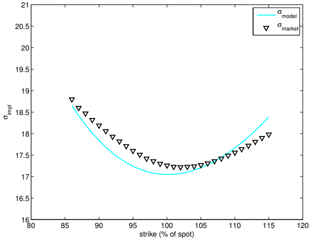
Dynamic SABR model - Calibration example



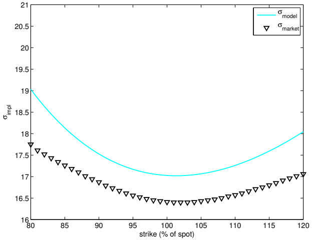
(e) 3 months maturity.



(f) 6 months maturity.



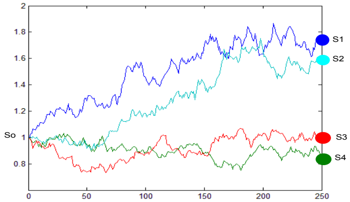
(g) 12 months maturity.



(h) 24 months maturity.

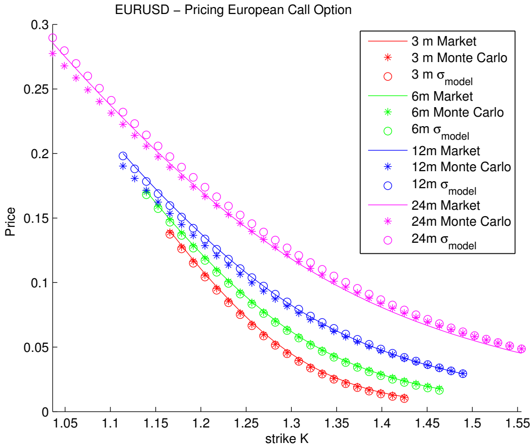
SABR pricing

- Monte Carlo:
  - huge number of forward and volatility paths
  - $V(S_0, K) = D(T)\mathbb{E}\big(V(S_T, K)\big)$

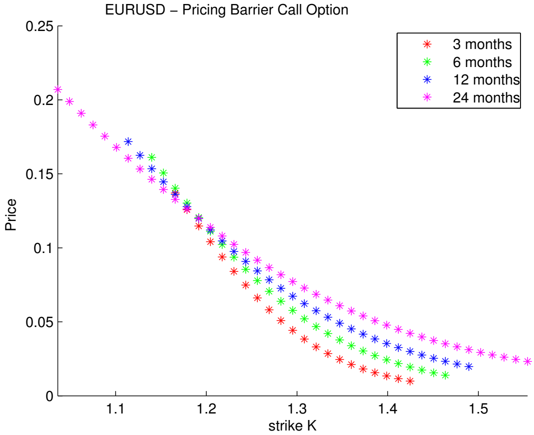


- Discretization schemes.
  - Euler.
  - Milstein.
  - log-Euler.
  - low-bias.
- Time step(Δt) or number of time steps.

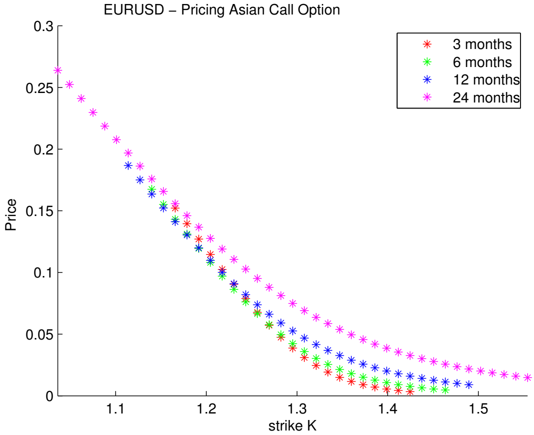
SABR pricing - European



SABR pricing - Barrier



SABR pricing - Asian



SABR Risk mesures

- Δ risk

$$\frac{\partial V}{\partial \hat{f}} = \frac{\partial BS}{\partial \hat{f}} + \frac{\partial BS}{\partial \sigma_B}\frac{\partial \sigma_B}{\partial \hat{f}}$$

- *Vega* risk

$$\frac{\partial V}{\partial \sigma_B} = \frac{\partial BS}{\partial \sigma_B}\frac{\partial \sigma_B}{\partial \alpha}$$

- *Vanna* risk

$$\frac{\partial V}{\partial \rho} = \frac{\partial BS}{\partial \sigma_B}\frac{\partial \sigma_B}{\partial \rho}$$

- *Volga* risk

$$\frac{\partial V}{\partial \nu} = \frac{\partial BS}{\partial \sigma_B}\frac{\partial \sigma_B}{\partial \nu}$$

References

José Luis Fernández, Ana María Ferreiro, José Antonio García-Rodríguez, Álvaro Leitao, José Germán López-Salas, and Carlos Vázquez.
Static an dynamic SABR stochastic volatility models: Calibration and option pricing using GPUs.
Mathematics and Computers in Simulation, 2013.

Patrick S. Hagan, Deep Kumar, Andrew S. Lesniewski, and Diana E. Woodward.
Managing smile risk.
Wilmott Magazine, 2002.

Jan Obloj.
Fine-tune your smile: Correction to Hagan et al, 2013.

Yasufumi Osajima.
The asymptotic expansion formula of implied volatility for dynamic SABR model and FX Hybrid model, 2007.

Questions

Thank you

Thanks
Gracias