Outline

1. Introduction
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3. Dynamic SABR model
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Introduction

- Since 70’s: Black-Scholes.
  - Standard option pricing method. Hypothesis:
    - The price follows lognormal distribution.
    - The volatility is constant.

- Models which modify the price distribution.
- Models which allow non-constant volatility.
  - Local volatility models: Dupire.
  - Stochastic volatility models: Heston or SABR.

(a) Smile
(b) Skew
Local vs. Stochastic volatility models

- LVM volatility is a function.
- Both capture smile well.
- Both can be used for pricing.
- LVM show an opposite dynamic.

- LVM problems with risk measures.
- SVM solve it. Volatility also follows a stochastic process.
SABR model

SABR model (Hagan et al. 2002)

\[ dF_t = \alpha_t F_t^\beta dW_t^1, \quad F_0 = \hat{f} \]
\[ d\alpha_t = \nu \alpha_t dW_t^2, \quad \alpha_0 = \alpha \]

- Forward, \( F_t = S_t e^{(r-q)(T-t)} \), where \( r \) is constant interest rate, \( q \) constant dividend yield and \( T \) maturity date.
- Volatility, \( \alpha_t \).
- \( dW_t^1 \) y \( dW_t^2 \), correlated geometric brownian motions:
  \[ dW_1 dW_2 = \rho dt \]
- Initial values: \( S_0 \) y \( \alpha \).
- Model parameters: \( \alpha, \beta, \nu \) and \( \rho \).
- S-tochastic A-pha B-eta R-ho model.
SABR model - Implied volatility

\[ \sigma_B(K, \hat{f}, T) = \frac{\alpha}{(K\hat{f})^{(1-\beta)/2}} \left[ 1 + \frac{(1-\beta)^2}{24} \ln^2 \left( \frac{\hat{f}}{K} \right) + \frac{(1-\beta)^4}{1920} \ln^4 \left( \frac{\hat{f}}{K} \right) + \cdots \right] \cdot \left( \frac{z}{x(z)} \right) \cdot \left[ 1 + \frac{(1-\beta)^2}{24} \frac{\alpha^2}{(K\hat{f})^{1-\beta}} + \frac{1}{4} \frac{\rho \beta \nu \alpha}{(K\hat{f})^{(1-\beta)/2}} + \frac{2 - 3 \rho^2}{24} \nu^2 \right] \cdot T + \cdots. \]

Note that the previous expression depends on the parameters \( K, \hat{f} \) and \( T \), also through the functions:

\[ z = \frac{\nu}{\alpha} (K\hat{f})^{(1-\beta)/2} \ln \left( \frac{\hat{f}}{K} \right), \]

and

\[ x(z) = \ln \left( \frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1 - \rho} \right). \]
SABR model - Obloj correction (2008)

\[ \sigma_B(K, \hat{f}, T) = \frac{1}{\left[ 1 + \frac{(1 - \beta)^2}{24} \ln^2 \left( \frac{\hat{f}}{K} \right) + \frac{(1 - \beta)^4}{1920} \ln^4 \left( \frac{\hat{f}}{K} \right) + \cdots \right]} \cdot \left( \frac{\nu \ln \left( \frac{\hat{f}}{K} \right)}{x(z)} \right) \cdot T + \cdots, \]

where the following new expression for \( z \) is considered:

\[ z = \frac{\nu \left( \hat{f}^{1-\beta} - K^{1-\beta} \right)}{\alpha(1 - \beta)}, \]

and \( x(z) \) is given by the same previous expression.

- The omitted terms can be neglected.
SABR model - Approx. implied volatility

\[ \sigma_B(K, \hat{f}, T) = \frac{1}{\omega} \left( 1 + A_1 \ln \left( \frac{K}{\hat{f}} \right) + A_2 \ln^2 \left( \frac{K}{\hat{f}} \right) + B T \right), \]

where the coefficients \( A_1, A_2 \) and \( B \) are given by

\[ A_1 = -\frac{1}{2} (1 - \beta - \rho \nu \omega), \]
\[ A_2 = \frac{1}{12} \left( (1 - \beta)^2 + 3((1 - \beta) - \rho \nu \omega) + (2 - 3\rho^2) \nu^2 \omega^2 \right), \]
\[ B = \frac{(1 - \beta)^2}{24} \frac{1}{\omega^2} + \frac{\beta \rho \nu}{4} \frac{1}{\omega} + \frac{2 - 3\rho^2}{24} \nu^2, \]

and the value of \( \omega \) is given by

\[ \omega = \frac{\hat{f}^{1-\beta}}{\alpha}. \]
(e) $\alpha > 0$, the volatility's reference level.

(f) $0 \leq \beta \leq 1$, the variance elasticity.

(g) $\nu > 0$, the volatility of the volatility.

(h) $-1 \leq \rho \leq 1$, the correlation coefficient.
The calibration process tries to obtain a set of model parameters that makes model values as close as possible to market ones, i.e.

\[ V_{\text{market}}(K_j, \hat{f}, T_i) \approx V_{\text{sabr}}(K_j, \hat{f}, T_i) \]

In order to achieve this target we must follow several steps:

- Prices or volatilities.
- Representative market data.
- Error measure.
- Cost function.
- Optimization algorithm.
- Fix parameters on beforehand.
- Calibrate and compare the obtained results.
SABR model - Calibration example

(i) 3 months maturity.

(j) 6 months maturity.

(k) 12 months maturity.

(l) 24 months maturity.
Figure: Market volatility surface.
SABR model - Drawback

(a) 3 months maturity.

(b) 6 months maturity.

(c) 12 months maturity.

(d) 24 months maturity.
Dynamic SABR model

\[
dF_t = \alpha_t F_t^\beta dW_t^1, \quad F_0 = \hat{f} \\
d\alpha_t = \nu(t)\alpha_t dW_t^2, \quad \alpha_0 = \alpha
\]

- Forward, \( F_t \).
- Volatility, \( \alpha_t \).
- \( dW_t^1 \) y \( dW_t^2 \), correlated geometric brownian motions:

\[
dW_1 dW_2 = \rho(t)dt
\]

- Initial values: \( S_0 \) y \( \alpha \).
- Model parameters: \( \alpha, \beta \) and ones that \( \nu(t) \) and \( \rho(t) \) can provide.
- Approximation of implied volatility provided by Osajima (2007).
Dynamic SABR model

dynamic SABR model

\[ dF_t = \alpha_t F_t^\beta dW_t^1, \quad F_0 = \hat{f} \]
\[ d\alpha_t = \nu(t) \alpha_t dW_t^2, \quad \alpha_0 = \alpha \]

- Forward, \( F_t \).
- Volatility, \( \alpha_t \).
- \( dW_t^1 \) y \( dW_t^2 \), correlated geometric brownian motions:

\[ dW_1 dW_2 = \rho(t) dt \]

- Inicial values: \( S_0 \) y \( \alpha \).
- Model parameters: \( \alpha, \beta \) and ones that \( \nu(t) \) and \( \rho(t) \) can provide.
- Approximation of implied volatility provided by Osajima (2007).
Dynamic SABR model - Approx. implied volatility

\[ \sigma_B(K, \hat{f}, T) = \frac{1}{\omega} \left( 1 + A_1(T) \ln \left( \frac{K}{\hat{f}} \right) + A_2(T) \ln^2 \left( \frac{K}{\hat{f}} \right) + B(T) T \right), \]

where

\[ A_1(T) = \frac{\beta - 1}{2} + \frac{\eta_1(T) \omega}{2}, \]
\[ A_2(T) = \frac{(1 - \beta)^2}{12} + \frac{1 - \beta - \eta_1(T) \omega}{4} + \frac{4 \nu_1^2(T) + 3 (\eta_2^2(T) - 3 \eta_1^2(T))}{24} \omega^2, \]
\[ B(T) = \frac{1}{\omega^2} \left( \frac{(1 - \beta)^2}{24} + \frac{\omega \beta \eta_1(T)}{4} + \frac{2 \nu_2^2(T) - 3 \eta_2^2(T)}{24} \omega^2 \right), \]

with

\[ \nu_1^2(T) = \frac{3}{T^3} \int_0^T (T - t)^2 \nu^2(t) dt, \quad \nu_2^2(T) = \frac{6}{T^3} \int_0^T (T - t) t \nu^2(t) dt, \]
\[ \eta_1(T) = \frac{2}{T^2} \int_0^T (T - t) \nu(t) \rho(t) dt, \quad \eta_2^2(T) = \frac{12}{T^4} \int_0^T \int_0^t \left( \int_0^s \nu(u) \rho(u) du \right)^2 ds dt. \]
Dynamic SABR model - $\rho(t)$ and $\nu(t)$ functions

- $\rho(t)$ and $\nu(t)$ have to be smaller for long terms ($t$ large) rather than for short terms ($t$ small).

<table>
<thead>
<tr>
<th>Constant</th>
<th>Piecewise</th>
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<tbody>
<tr>
<td>$\rho(t) = \rho_0$</td>
<td>$\rho(t) = \rho_0, t \leq T_0$</td>
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<tr>
<td>$\nu(t) = \nu_0$</td>
<td>$\nu(t) = \nu_0, t \leq T_0$</td>
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<td></td>
<td>$\alpha, \beta, \rho_0, \nu_0$, SABR model.</td>
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<th>Classical</th>
<th>General</th>
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<tr>
<td>$\rho(t) = \rho_0 e^{-at}$</td>
<td>$\rho(t) = (\rho_0 + q_\rho t)e^{-at} + d_\rho$</td>
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<tr>
<td>$\nu(t) = \nu_0 e^{-bt}$</td>
<td>$\nu(t) = (\nu_0 + q_\nu t)e^{-bt} + d_\nu$</td>
</tr>
<tr>
<td>$\alpha, \beta, \rho_0, \nu_0, a$ and $b$</td>
<td>$\alpha, \beta, \rho_0, \nu_0, a, b, d_\rho, d_\nu, q_\rho$ and $q_\nu$.</td>
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Dynamic SABR model - Classical choice

\[
\nu_1^2(T) = \frac{6\nu_0^2}{(2bT)^3} \left[ \frac{(2bT)^2}{2} - 2bT + 1 - e^{-2bT} \right],
\]

\[
\nu_2^2(T) = \frac{6\nu_0^2}{(2bT)^3} \left[ 2(e^{-2bT} - 1) + 2bT(e^{-2bT} + 1) \right],
\]

\[
\eta_1(T) = \frac{2\nu_0\rho_0}{T^2(a+b)^2} \left[ e^{-(a+b)T} - (1 - (a+b)T) \right],
\]

\[
\eta_2^2(T) = \frac{3\nu_0^2\rho_0^2}{T^4(a+b)^4} \left[ 1 - 8e^{-(a+b)T} + \left( 7 + 2(a+b)T \left( -3 + (a+b)T \right) \right) \right].
\]
Dynamic SABR model - Calibration example

(e) 3 months maturity.

(f) 6 months maturity.

(g) 12 months maturity.

(h) 24 months maturity.
SABR pricing

- Monte Carlo:
  - huge number of forward and volatility paths
  - $V(S_0, K) = D(T) \mathbb{E}(V(S_T, K))$

- Discretization schemes.
  - Euler.
  - Milstein.
  - log-Euler.
  - low-bias.

- Time step($\Delta t$) or number of time steps.
SABR pricing - European

EURUSD – Pricing European Call Option

- 3 m Market
- 3 m Monte Carlo
- 3 m $\sigma_{\text{model}}$
- 6 m Market
- 6 m Monte Carlo
- 6 m $\sigma_{\text{model}}$
- 12 m Market
- 12 m Monte Carlo
- 12 m $\sigma_{\text{model}}$
- 24 m Market
- 24 m Monte Carlo
- 24 m $\sigma_{\text{model}}$

Price vs. strike K

Álvaro Leitao (Lecture group, CWI)
SABR pricing - Barrier

EURUSD – Pricing Barrier Call Option

- 3 months
- 6 months
- 12 months
- 24 months

strike K
Price

3 months
6 months
12 months
24 months
SABR pricing - Asian

EURUSD – Pricing Asian Call Option

- 3 months
- 6 months
- 12 months
- 24 months

strike K
Price

SABR model

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SABR Risk measures

- Δ risk
  \[ \frac{\partial V}{\partial \hat{f}} = \frac{\partial BS}{\partial \hat{f}} + \frac{\partial BS}{\partial \sigma_B} \frac{\partial \sigma_B}{\partial \hat{f}} \]

- Vega risk
  \[ \frac{\partial V}{\partial \sigma_B} = \frac{\partial BS}{\partial \sigma_B} \frac{\partial \sigma_B}{\partial \alpha} \]

- Vanna risk
  \[ \frac{\partial V}{\partial \rho} = \frac{\partial BS}{\partial \sigma_B} \frac{\partial \sigma_B}{\partial \rho} \]

- Volga risk
  \[ \frac{\partial V}{\partial \nu} = \frac{\partial BS}{\partial \sigma_B} \frac{\partial \sigma_B}{\partial \nu} \]
References

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Questions
Thank you

Thanks

Gracias