The data-driven COS method
Application to option pricing

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Motivation

- Combine the ideas behind the COS method and Monte Carlo simulation.
- Preserving the individual advantages and overcoming the particular disadvantages.
- Make the COS method more generally and directly applicable and more flexible.
- Improve the convergence of the Monte Carlo method.
Definitions

Option

A contract that offers the buyer the right, but not the obligation, to buy (call) or sell (put) a financial asset at an agreed-upon price (the strike price) during a certain period of time or on a specific date (exercise date).

Investopedia.

Option price

The fair value to enter in the option contract. In other words, the (discounted) expected value of the contract.

\[ V_\tau = D_\tau \mathbb{E} [v(S(\tau))] \]

where \( v(\cdot) \) is the payoff function, \( S(\tau) \) the future value of an underlying asset, \( S(t) \), and \( D_\tau \) the discount factor.
### Pricing techniques

- Stochastic process, $S(t)$: Stochastic differential equation (SDE).
- Simulation: Monte Carlo method.
- Fourier-based methods.
- PDEs: Feynman-Kac theorem.

### Types of options - Exercise time

- **European**: End of the contract, $\tau = T$.
- **Early-exercise**: American ($\tau \in [t, T]$) or Bermudan ($\tau \in \{t1, \ldots, tM\}$).
- Many others: Asian, barrier, ... 

### Types of options - Payoff

- **Vanilla**: $[c(S(\tau) - K)]^+$, call ($c = 1$) or put ($c = -1$).
- Many others: Digital, Gap, ...
The COS method

- A lot of work behind: [FO08], [FO09], etc.
- Fourier-based method to price options.
- Point of departure is risk-neutral valuation formula:

\[ v(x, t) = e^{-r(\tau-t)} \mathbb{E}[v(y, \tau)|x] = e^{-r(\tau-t)} \int_{\mathbb{R}} v(y, \tau)f(y|x)dy, \]

where \( r \) is the risk-free rate and \( f(y|x) \) is the density of the underlying process. Typically, we have:

\[ x := \log \left( \frac{S(t)}{K} \right) \quad \text{and} \quad y := \log \left( \frac{S(t + \tau)}{K} \right). \]

- \( f(y|x) \) is unknown in most of cases.
- However, characteristic function available for many models.
- Exploit the relation between the density and the characteristic function (Fourier pair).
For a function supported on $[0, \pi]$, the cosine series expansion reads

$$f(\theta) = \sum_{k=0}^{\infty} A_k \cdot \cos (k\theta).$$

where $\sum'$ indicates that the first term is weighted by one-half.

Other finite interval $[a, b]$, change of variables:

$$\theta := \pi \frac{x - a}{b - a}.$$

How to compute the support $[a, b]$ for a particular problem is crucial.

The COS method relies on a *cumulant* approach.
Pricing European options with the COS method

- The cosine series expansion of \( f(y|x) \) in the support \([a, b]\) is

\[
f(y|x) = \sum_{k=0}^{\infty} A_k(x) \cdot \cos \left( k\pi \frac{y - a}{b - a} \right).
\]

- The option value, \( v(x, t) \) with \( \tau = T \), can be then approximated by

\[
v(x, t) \approx e^{-r(T-t)} \int_{a}^{b} v(y, T) \sum_{k=0}^{\infty} A_k(x) \cos \left( k\pi \frac{y - a}{b - a} \right) dy.
\]

- Interchanging sum and integration, and introducing the definition

\[
V_k := \frac{2}{b - a} \int_{a}^{b} v(y, T) \cos \left( k\pi \frac{y - a}{b - a} \right) dy,
\]

an approximated pricing formula is obtained (after series truncation)

\[
v(x, t) \approx \frac{1}{2}(b - a)e^{-r(T-t)} \sum_{k=0}^{N-1} A_k(x) V_k.
\]
Pricing European options with the COS method

- The $A_k(x)$ expansion coefficients are
  \[
  A_k(x) = \frac{2}{b-a} \int_a^b f(y|x) \cos \left( k\pi \frac{y-a}{b-a} \right) \, dy.
  \]

- By employing the Fourier transform properties and based on the characteristic function, $\phi(u;x)$, associated to $f(y|x)$:
  \[
  A_k(x) \approx \frac{2}{b-a} \mathcal{R} \left\{ \phi \left( \frac{k\pi}{b-a}; x \right) \cdot \exp \left( -i \frac{k\pi a}{b-a} \right) \right\}.
  \]

- The COS pricing formula for European options
  \[
  v(x, t) \approx e^{-r(T-t)} \sum_{k=0}^{N-1} \mathcal{R} \left\{ \phi \left( \frac{k\pi}{b-a}; x \right) \cdot \exp \left( -i \frac{k\pi a}{b-a} \right) \right\} V_k.
  \]

- The $V_k$ coefficients are known for many types of payoffs.
A Bermudan option can be exercised at a set of predefined dates.

The price is computed by using the risk-neutral valuation formula.

With $M$ exercise dates $t_1 < \cdots < t_M = T$ and with $\Delta t = t_m - t_{m-1}$, the pricing formula for Bermudan option then reads

$$c(x, t_{m-1}) = e^{-\Delta t} \int_{\mathbb{R}} v(y, t_m) f(y|x) \, dy,$$

$$v(x, t_{m-1}) = \max (g(x, t_{m-1}), c(x, t_{m-1})),$$

applied recursively, starting in $t_m = T$ until $t_2$, and followed by

$$v(x, t_0) = e^{-\Delta t} \int_{\mathbb{R}} v(y, t_1) f(y|x) \, dy.$$

The functions $v(x, t)$, $c(x, t)$ and $g(x, t)$ are the option value, the continuation value and the payoff value at time $t$, respectively.
Following a similar procedure as in the case of the European options, the continuation value and the option value can be approximated as

\[ c(x, t_{m-1}) \approx e^{-\Delta t} \sum_{k=0}^{N-1} \mathcal{R} \left\{ \phi \left( \frac{k \pi}{b - a}; x \right) \cdot \exp \left( -i \frac{k \pi a}{b - a} \right) \right\} V_k(t_m), \]

and

\[ v(x, t_0) \approx e^{-\Delta t} \sum_{k=0}^{N-1} \mathcal{R} \left\{ \phi \left( \frac{k \pi}{b - a}; x \right) \cdot \exp \left( -i \frac{k \pi a}{b - a} \right) \right\} V_k(t_1), \]

where, again, \( \phi(u; x) \) is the characteristic function.

- Pricing a Bermudan option is reduced to the computation of \( V_k(t_1) \).
- The coefficients \( V_k \) at any time \( t_m \) can be defined by

\[ V_k(t_m) := \frac{2}{b - a} \int_a^b v(y, t_m) \cos \left( k \pi \frac{y - a}{b - a} \right) dy. \]
“Learning” densities from data

- **Statistical learning theory**: deals with the problem of finding a predictive function based on data. Wikipedia.
- We follow the analysis about the problem of density estimation proposed by Vapnik in [Vap98].
- Given independent and identical distributed samples $X_1, X_2, \ldots, X_n$.
- By definition, density $f(x)$ is related to the *cumulative distribution function*, $F(x)$, by means of the expression

$$\int_{-\infty}^{x} f(y)\,dy = F(x).$$

- $F(x)$ can be very accurately approximated by the empirical equivalent

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \eta(x - X_i),$$

where $\eta(\cdot)$ is the step-function. Convergence $O(1/\sqrt{n})$. 
Regularization approach

- The previous equation can be rewritten as a linear operator equation

\[ Af = F \approx F_n, \]

where the operator \( Az = \int_{-\infty}^{\infty} z \, dz \).


- Given a lower semi-continuous functional \( W(f) \) such that:
  - Solution of \( Af = F_n \) belongs to \( \mathcal{D} \), the domain of definition of \( W(f) \).
  - The functional \( W(f) \) takes real non-negative values in \( \mathcal{D} \).
  - The set \( \mathcal{M}_c = \{ f : W(f) \leq c \} \) is compact in \( \mathcal{H} \) (the space where the solution exits and is unique).

- Then we can construct the functional

\[ R_{\gamma_n}(f, F_n) = L^2_{\mathcal{H}}(Af, F_n) + \gamma_n W(f), \]

where \( L_{\mathcal{H}} \) is a metric of the space \( \mathcal{H} \) (loss function) and \( \gamma_n \) is the parameter of regularization satisfying that \( \gamma_n \to 0 \) as \( n \to \infty \).

- Under these conditions, a function \( f_n \) minimizing the functional converges almost surely to the desired one.
The regularization approach and the COS method

- Assuming $f(x)$ belongs to the functions whose $p$-th derivatives belong to $L_2(0, \pi)$. We consider the risk functional as the form

$$R_{\gamma_n}(f, F_n) = \int_0^{\pi} \left( \int_0^x f(y) dy - F_n(x) \right)^2 dx + \gamma_n \int_0^{\pi} (f^{(p)}(x))^2 dx.$$ 

- Assuming the solution is in the form (as in the COS method)

$$f_n(\theta) = \sum_{k=0}^{\infty} \hat{A}_k \cos(k\theta),$$

where $\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_{k-1}, \ldots$ are the expansion coefficients.

- Plugging the expansion in the risk functional, it can be proved that the minimum of $R_{\gamma_n}(f_n, F_n)$ is reached when

$$\hat{A}_k = \frac{2}{\pi} \cdot \frac{1}{1 + \gamma_n k^{2(p+1)}} \sum_{i=1}^{n} \cos(k\theta_i), \quad k = 0, 1, 2, \ldots.$$ 

where, again, $n$ is the number of available samples.
Choice of the free parameters: $\gamma_n$ and $p$

- The choice of optimal values of $\gamma_n$ and $p$ is crucial in terms of accuracy and efficiency.
- There is no rule or procedure to obtain an optimal $p$.
- As a rule of thumb, $p = 0$ seems to be the most appropriate value.
- Fixing $p$, we rely on the computation of an optimal $\gamma_n$.

![Parameter $p$ analysis](image)

Figure: Parameter $p$ analysis.
Choice of $\gamma_n$

- For the regularization parameter $\gamma_n$, a rule that ensures asymptotic convergence

$$\gamma_n = \frac{\log \log n}{n}.$$ 

- But it is not the optimal value of $\gamma_n$, i.e. the one which provides the fastest convergence in practical situations.
Choice of $\gamma_n$

- Exploit the relation between the empirical and real (unknown) CDFs.
- This relation can be modeled by statistical laws or statistics: Kolmogorov-Smirnov, Anderson-Darling, Smirnov-Cramér–von Mises.
- Preferable: a measure of the distance between the $F_n(x)$ and $F(x)$ follows a known distribution.
- We have chosen Smirnov-Cramér–von Mises (SCvM):

$$\omega^2 = n \int_{\mathbb{R}} (F(x) - F_n(x))^2 \, dF(x).$$

- Assume we have an approximation, $F_{\gamma_n}$ (which depends on $\gamma_n$).
- An almost optimal $\gamma_n$ is computed by solving the equation

$$\sum_{i=1}^{n} \left( F_{\gamma_n}(\bar{X}_i) - \frac{i - 0.5}{n} \right)^2 = m_S - \frac{i}{12n},$$

where $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n$ is the ordered array of samples $X_1, X_2, \ldots, X_n$ and $m_S$ the mean of the $\omega^2$. 
Influence of $\gamma_n$

- To assess the impact of $\gamma_n$: *Mean integrated Squared Error* (MiSE):

$$\mathbb{E} \left[ \| f_n - f \|_2^2 \right] = \mathbb{E} \left[ \int_{\mathbb{R}} (f_n(x) - f(x))^2 \, dx \right].$$

- A formula for the MiSE formula is derived in our context:

$$\text{MiSE} = \frac{1}{n} \sum_{k=1}^{N} \frac{1}{(1 + \gamma_n k^2(p+1))^2} \left( \frac{1}{2} + \frac{1}{2} A_{2k} - A_k^2 \right) + \sum_{k=N+1}^{\infty} A_k^2.$$

- Two main aspects influenced $\gamma_n$: accuracy in $n$ and stability in $N$. 
Influence of $\gamma_n$

(a) Convergence in terms of $n$

(b) Convergence in terms of $N$

Figure: Influence of $\gamma_n$: .
Optimal number of terms $N$

- Try to find a *minimum optimal* value of $N$.
- $N$ considerably affects the performance.
- We wish to avoid the computation of any $\hat{A}_k$.
- We define a proxy for the MiSE and follow:

$$\text{MiSE} \approx \frac{1}{n} \sum_{k=1}^{N} \frac{1}{2} \left(1 + \gamma_n k^2(p+1)\right)^2.$$

![Figure: MiSE proxy.](image)
**Optimal number of terms $N$**

**Data:** $n$, $\gamma_n$

$N_{min} = 5$

$N_{max} = \infty$

$\epsilon = \frac{1}{\sqrt{n}}$

$\text{MiSE}_{prev} = \infty$

for $N = N_{min} : N_{max}$ do

$\text{MiSE}_N = \frac{1}{n} \sum_{k=1}^{N} \left( 1 + \gamma_n k^2 (p+1) \right)^2$

$\epsilon_N = \frac{|\text{MiSE}_N - \text{MiSE}_{prev}|}{|\text{MiSE}_N|}$

if $\epsilon_N > \epsilon$ then

$N_{op} = N$

else

Break

$\text{MiSE}_{prev} = \text{MiSE}_N$

**Figure:** Almost optimal $N$. 
Employ the regularization approach for PDF estimation in the COS framework.

In both, the PDF is assumed to be in the form of a cosine series expansion.

The minimum of the functional is in terms of the expansion coefficients.

Take advantage of the COS machinery: pricing options.

The samples are generated by Monte Carlo (risk-neutral measure).
The data-driven COS method - European options

- Key idea: the data-based $\hat{A}_k$ approximates the $A_k(x)$ in COS method.
- Let assume that we have the samples $S_i(t)$.
- As in COS method, a logarithmic transformation is made

$$Y_j := \log \left( \frac{S_j(T)}{K} \right).$$

- Due to the solution is defined in $(0, \pi)$, we further transform the samples as

$$\theta_i := \pi \frac{Y_i - a}{b - a}.$$  

where the quantities $a$ and $b$ are defined as

$$a := \min_{1 \leq j \leq n} (Y_j), \quad b := \max_{1 \leq j \leq n} (Y_j).$$

- The samples, $Y_i$, must intrinsically consider the dependency on $x$ of the function-like coefficients, $A_k(x)$.
- This is fulfilled when generated by the Monte Carlo method.
The data-driven COS method - European options

- New expression for the data-driven coefficients, $\hat{A}_k$:

$$\hat{A}_k = \frac{2}{b-a} \cdot \frac{\frac{1}{n} \sum_{i=1}^{n} \cos \left( k\pi \frac{Y_i-a}{b-a} \right)}{1 + \gamma_n k^2(p+1)}, \quad k = 1, 2, \ldots.$$  

- By substituting the $A_k(x)$ in the COS formula by the $\hat{A}_k$ coefficients, we obtain the ddCOS pricing formula for European options

$$\hat{\nu}(x, t) = \frac{1}{2} (b-a)e^{-r(T-t)} \sum_{k=0}^{N-1} \hat{A}_k V_k,$$

$$= e^{-r(T-t)} \sum_{k=0}^{N-1} \frac{\frac{1}{n} \sum_{i=1}^{n} \cos \left( k\pi \frac{Y_i-a}{b-a} \right)}{1 + \gamma_n k^2(p+1)} \cdot V_k.$$
The ddCOS method vs. Monte Carlo

Figure: Convergence GBM.
The ddCOS method vs. Monte Carlo

- With more involved models: Jump-diffusion Merton model.

Figure: Convergence Merton.
The ddCOS method vs. Monte Carlo

- Almost optimal $\gamma_n$ (SCvM) does not provide better results.
- Natural question: is the ddCOS method worth to use?
- Other words: is it better in terms of computational cost?

<table>
<thead>
<tr>
<th>RE</th>
<th>$&lt; 10^{-1}$</th>
<th>$&lt; 10^{-2}$</th>
<th>$&lt; 10^{-3}$</th>
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<tr>
<td>GBM</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>MC</td>
<td>0.0095($10^3$)</td>
<td>0.0147($10^5$)</td>
<td>0.6721($10^7$)</td>
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<td>ddCOS</td>
<td>0.0256($10^1$)</td>
<td>0.0258($10^3$)</td>
<td>0.2696($10^5$)</td>
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<td>Speedup</td>
<td>$\times 0.37$</td>
<td>$\times 0.57$</td>
<td>$\times 2.49$</td>
</tr>
<tr>
<td>Merton</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MC</td>
<td>0.0396($10^3$)</td>
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<td>Speedup</td>
<td>$\times 0.75$</td>
<td>$\times 2.36$</td>
<td>$\times 35.24$</td>
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Table: Time(s) vs. accuracy. In parentheses, the number of required samples, $n$. 

Speedup $\times 0.37 < 10^{-1}$, $\times 0.57 < 10^{-2}$, $\times 2.49 < 10^{-3}$
### The ddCOS method - Pricing European options

<table>
<thead>
<tr>
<th>( K (% \text{ of } S_0) )</th>
<th>80%</th>
<th>90%</th>
<th>100%</th>
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<td>( 5.7344 \times 10^{-5} )</td>
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</table>

**Table:** GBM. European call, \( S_0 = 100 \), \( T = 1 \), \( r = 0.05 \), \( \sigma = 0.15 \).

<table>
<thead>
<tr>
<th>( K (% \text{ of } S_0) )</th>
<th>80%</th>
<th>90%</th>
<th>100%</th>
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<tr>
<td>Merton</td>
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**Table:** Merton. European call, \( S_0 = 100 \), \( T = 5 \), \( r = 0.1 \), \( \sigma = 0.3 \), \( \lambda = 3 \), \( \mu_J = -0.2 \), \( \sigma_J = 0.2 \).
## The ddCOS method - Pricing European options

### CEV

<table>
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<tr>
<th>$K$ (% of $S_0$)</th>
<th>80%</th>
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<td>26.3220</td>
<td>18.1347</td>
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<td>$2.3440 \times 10^{-6}$</td>
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**Table:** CEV. European call, $S_0 = 100$, $T = 2$, $r = 0.1$, $\sigma = 0.3$, $\beta = 0.5$.

### SABR

<table>
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<tr>
<th>$K$ (% of $S_0$)</th>
<th>80%</th>
<th>90%</th>
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**Table:** SABR. European call, $S_0 = 100$, $T = 1$, $r = 0.05$, $\sigma_0 = 0.15$, $\alpha = 0.3$, $\beta = 0.8$, $\rho = -0.8$. 
Pricing Bermudan options under the ddCOS is more involved: several exercise times, $t_m$.

The characteristic function now appears in the computation of both, the continuation value and the final option value.

The “conditionalty” in the samples is not straightforward since the current state at $t_m$ is conditional on the previous one at $t_{m-1}$ (not on the initial state $S_0$).

For pricing Bermudan option, the state variables $x$ and $y$ are defined

$$x := \log \left( \frac{S(t_{m-1})}{K} \right) \quad \text{and} \quad y := \log \left( \frac{S(t_m)}{K} \right).$$
We propose the following approximation

\[ f(y|x) \overset{d}{=} f(y - x) \]

\[ \overset{d}{=} f \left( \log \left( \frac{S(t_m)}{K} \right) - \log \left( \frac{S(t_{m-1})}{K} \right) \right) \]

\[ \overset{d}{=} f \left( \log \left( \frac{S(t_m)}{S(t_{m-1})} \right) \right) \overset{d}{=} f \left( \log \left( \frac{S(t_1)}{S(t_0)} \right) \right). \]

where \( \overset{d}{=} \) indicates equality in the distribution sense.

If we consider a particular realization, \( x' \), then

\[ f(y|x = x') \overset{d}{=} f \left( \log \left( \frac{S(t_1)}{S(t_0)} \right) \right) + x'. \]
Schematic representation of the idea:

Figure: Approximation to the continuation value computation.
According to that, we define the samples as

\[ Z_j = \log \left( \frac{S_j(t_1)}{S(t_0)} \right), \]

By applying the ddCOS method we have

\[ \hat{A}_k(x) = \frac{\frac{1}{n} \sum_{j=1}^{n} \cos \left( k\pi \frac{(Z_j+x)-a}{b-a} \right)}{1 + \gamma_n k^2(p+1)}. \]

Then, the data-based expression for the continuation value

\[ \hat{c}(x, t_{m-1}) = \exp(-\Delta t) \sum_{k=0}^{N-1} \hat{B}_k(x) V_k(t_m) \]

\[ = \exp(-\Delta t) \sum_{k=0}^{N-1} \frac{\frac{1}{n} \sum_{i=1}^{n} \cos \left( k\pi \frac{(Z_i(t_1)+x)-a}{b-a} \right)}{1 + \gamma_n k^2(p+1)} \cdot V_k(t_m). \]
### The ddCOS method - Pricing Bermudan options

#### Table: GBM. Bermudan put, \( S_0 = 100, \ T = 2, \ r = 0.05, \ \sigma = 0.2. \)

<table>
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<th>( K(% \text{ of } S_0) )</th>
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<tbody>
<tr>
<td>COS</td>
<td>1.6413</td>
<td>3.8766</td>
<td>7.6122</td>
<td>13.0919</td>
<td>20.3759</td>
</tr>
<tr>
<td>ddCOS</td>
<td>1.6400</td>
<td>3.8721</td>
<td>7.6011</td>
<td>13.0756</td>
<td>20.3730</td>
</tr>
<tr>
<td>MSE</td>
<td>( 8.3930 \times 10^{-5} )</td>
<td>( 8.3930 \times 10^{-5} )</td>
<td>( 8.3930 \times 10^{-5} )</td>
<td>( 8.3930 \times 10^{-5} )</td>
<td>( 8.3930 \times 10^{-5} )</td>
</tr>
</tbody>
</table>

#### Table: Merton. Bermudan put, \( S_0 = 100, \ T = 2, \ r = 0.05, \ \sigma = 0.2, \ \lambda = 3, \ \mu_J = -0.2, \ \sigma_J = 0.2. \)

<table>
<thead>
<tr>
<th>( K(% \text{ of } S_0) )</th>
<th>80%</th>
<th>90%</th>
<th>100%</th>
<th>110%</th>
<th>120%</th>
</tr>
</thead>
<tbody>
<tr>
<td>COS</td>
<td>13.2743</td>
<td>17.8149</td>
<td>22.9726</td>
<td>28.7044</td>
<td>34.9664</td>
</tr>
<tr>
<td>ddCOS</td>
<td>13.1908</td>
<td>17.7128</td>
<td>22.8526</td>
<td>28.5668</td>
<td>34.8134</td>
</tr>
<tr>
<td>MSE</td>
<td>( 1.4828 \times 10^{-2} )</td>
<td>( 1.4828 \times 10^{-2} )</td>
<td>( 1.4828 \times 10^{-2} )</td>
<td>( 1.4828 \times 10^{-2} )</td>
<td>( 1.4828 \times 10^{-2} )</td>
</tr>
</tbody>
</table>
Conclusions

- We have combined a density estimation procedure with the COS method.
- Resulting in a simple and very efficient technique for option pricing.
- The ddCOS method improves the convergence w.r.t. Monte Carlo.
- For high accuracy, faster than Monte Carlo.
- Ongoing work:
  - Case where the use of $\gamma_n$ by SCvM is justified.
  - “Variance reduction” for the ddCOS method.
  - Bermudan approximation.
  - 2D extension.
Fang Fang and Cornelis W. Oosterlee.

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Vladimir N. Vapnik.
Suggestions, comments & questions

Thank you for your attention