The data-driven COS method

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Outline

The COS method

- Well known and established method: [FO08], [FO09], etc.
- Fourier-based method to price financial options.
- Starting point is risk-neutral valuation formula:

$$v(x,t) = \mathrm{e}^{-r(T-t)}\mathbb{E}\left[v(y,T)|x\right] = \mathrm{e}^{-r(T-t)}\int_{\mathbb{R}} v(y,T)f(y|x)\mathrm{d}y,$$

  where r is the risk-free rate and $f(y|x)$ is the density of the underlying process. Typically, we have:

$$x := \log\left(\frac{S(0)}{K}\right) \quad \text{and} \quad y := \log\left(\frac{S(T)}{K}\right),$$

- $f(y|x)$ is unknown in most of cases.
- However, characteristic function available for many models.
- Exploit the relation between the density and the characteristic function (Fourier pair).

The COS method - European options

- $f(y|x)$ is approximated, on a finite interval $[a,b]$, by a cosine series

$$f(y|x) = \frac{1}{b-a}\left(A_0 + 2\sum_{k=1}^{\infty} A_k(x)\cdot\cos\left(k\pi\frac{y-a}{b-a}\right)\right),$$

$$A_0 = 1,\;\; A_k(x) = \int_a^b f(y|x)\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y, \quad k = 1,2,\ldots.$$

- Interchanging the summation and integration and introducing the definition

$$V_k := \frac{2}{b-a}\int_a^b v(y,T)\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y,$$

we find that the option value is given by

$$v(x,t) \approx \mathrm{e}^{-r(T-t)}{\sum_{k=0}^{\infty}}' A_k(x)V_k,$$

where $'$ indicates that the first term is divided by two.

Pricing European options with the COS method

- Coefficients A_k can be computed from the ChF.
- Coefficients V_k are known analytically (for many types of options).
- Closed-form expressions for the option Greeks Δ and Γ

$$\Delta = \frac{\partial v(x,t)}{\partial S} = \frac{1}{S(0)}\frac{\partial v(x,t)}{\partial x} \approx \exp(-r(T-t)) {\sum_{k=0}^{\infty}}' \frac{\partial A_k(x)}{\partial x}\frac{V_k}{S(0)},$$

$$\Gamma = \frac{\partial^2 v(x,t)}{\partial S^2} = \approx \exp(-r(T-t)) {\sum_{k=0}^{\infty}}' \left(-\frac{\partial A_k(x)}{\partial x)} + \frac{\partial^2 A_k(x)}{\partial x^2)}\right)\frac{V_k}{S^2(0)}$$

- Due to the rapid decay of the coefficients, $v(x,t)$, Δ and Γ can be approximated with high accuracy by truncating to N terms.

"Learning" densities

- *Statistical learning theory*: deals with the problem of finding a predictive function based on data.
- We follow the analysis about the problem of density estimation proposed by Vapnik in [Vap98].
- Given independent and identically distributed samples $X_1, X_2, \ldots, X_n$.
- By definition, density $f(x)$ is related to the *cumulative distribution function*, $F(x)$, by means of the expression

$$\int_{-\infty}^{x} f(y)\mathrm{d}y = F(x).$$

- Function $F(x)$ is approximated by the empirical approximation

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} \eta(x - X_i),$$

where $\eta(\cdot)$ is the step-function. Convergence $\mathcal{O}(1/\sqrt{n})$.

Regularization approach

- The previous equation can be rewritten as a linear operator equation

$$Cf = F \approx F_n,$$

  where the operator $Ch := \int_{-\infty}^{x} h(z)\mathrm{d}z$.
- Stochastic ill-posed problem. *Regularization method* (Vapnik).
- Given a lower semi-continuous functional $W(f)$ such that:
  - Solution of $Cf = F_n$ belongs to $\mathcal{D}$, the domain of definition of $W(f)$.
  - The functional $W(f)$ takes real non-negative values in $\mathcal{D}$.
  - The set $\mathcal{M}_c = \{f : W(f) \leq c\}$ is compact in $\mathcal{H}$ (the space where the solution exits and is unique).
- Then we can construct the functional

$$R_{\gamma_n}(f, F_n) = L^2_{\mathcal{H}}(Cf, F_n) + \gamma_n W(f),$$

  where $L_{\mathcal{H}}$ is a metric of the space $\mathcal{H}$ (loss function) and γ_n is the parameter of regularization satisfying that $\gamma_n \to 0$ as $n \to \infty$.
- Under these conditions, a function f_n minimizing the functional converges almost surely to the desired one.

Regularization and Fourier-based density estimators

- Assume $f(x)$ belongs to the functions whose p-th derivatives belong to $L_2(0,\pi)$, the kernel $\mathcal{K}(z-x)$ and

$$W(f) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{K}(z-x) f(x) \mathrm{d}x \right)^2 \mathrm{d}z,$$

- The risk functional

$$R_{\gamma_n}(f, F_n) = \int_{\mathbb{R}} \left(\int_0^x f(y)\mathrm{d}y - F_n(x) \right)^2 \mathrm{d}x + \gamma_n \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{K}(z-x) f(x) \mathrm{d}x \right)^2 \mathrm{d}z.$$

- Denoting by $\hat{f}(u)$, $\hat{F}_n(u)$ and $\hat{\mathcal{K}}(u)$ the Fourier transforms, by definition

$$\begin{aligned} \hat{F}_n(u) &= \frac{1}{2\pi} \int_{\mathbb{R}} F_n(x) \mathrm{e}^{-iux} \mathrm{d}x \\ &= \frac{1}{2n\pi} \int_{\mathbb{R}} \sum_{j=1}^{n} \eta(x - X_j) \mathrm{e}^{-iux} \mathrm{d}x = \frac{1}{n} \sum_{j=1}^{n} \frac{\exp(-iuX_j)}{iu}, \end{aligned}$$

where $i = \sqrt{-1}$ is the imaginary unit.

Regularization and Fourier-based density estimators

- By employing the *convolution theorem* and *Parseval's identity*

$$R_{\gamma_n}(f, F_n) = \left\| \frac{\hat{f}(u) - \frac{1}{n}\sum_{j=1}^{n} \exp(-iuX_j)}{iu} \right\|_{L_2}^2 + \gamma_n \left\| \hat{\mathcal{K}}(u)\hat{f}(u) \right\|_{L_2}^2 .$$

- The condition to minimize $R_{\gamma_n}(f, F_n)$ is given by,

$$\frac{\hat{f}(u)}{u^2} - \frac{1}{nu^2}\sum_{j=1}^{n} \exp(-iuX_j) + \gamma_n \hat{\mathcal{K}}(u)\hat{\mathcal{K}}(-u)\hat{f}(u) = 0,$$

which gives us,

$$\hat{f}_n(u) = \left(\frac{1}{1 + \gamma_n u^2 \hat{\mathcal{K}}(u)\hat{\mathcal{K}}(-u)} \right) \frac{1}{n} \sum_{j=1}^{n} \exp(-iuX_j).$$

Regularization and Fourier-based density estimators

- $\mathcal{K}(x) = \delta^{(p)}(x)$, and the desired PDF, $f(x)$ and its p-th derivative ($p \geq 0$) belongs to $L_2(0,\pi)$, the risk functional becomes

$$R_{\gamma_n}(f, F_n) = \int_0^\pi \left(\int_0^x f(y)\mathrm{d}y - F_n(x) \right)^2 \mathrm{d}x + \gamma_n \int_0^\pi \left(f^{(p)}(x) \right)^2 \mathrm{d}x.$$

- Given *orthonormal functions*, $\psi_1(\theta), \ldots, \psi_k(\theta), \ldots$

$$f_n(\theta) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \tilde{A}_k \psi_k(\theta),$$

  with $\tilde{A}_0, \tilde{A}_1, \ldots, \tilde{A}_k, \ldots$ expansion coefficients, $\tilde{A}_k =< f_n, \psi_k >$.
- The coefficients $\tilde{A}_k$ cannot be directly computed from f_n, but

$$\begin{aligned}\tilde{A}_k &=< f_n, \psi_k >=< \hat{f}_n, \hat{\psi}_k > \\ &= \int_0^\pi \left(\left(\frac{1}{1+\gamma_n u^2 \hat{\mathcal{K}}(u)\hat{\mathcal{K}}(-u)} \right) \frac{1}{n} \sum_{j=1}^{n} \exp(-iu\theta_j) \right) \cdot \hat{\psi}_k(u)\mathrm{d}u.\end{aligned}$$

Regularization and Fourier-based density estimators

- Using cosine series expansions, i.e., $\psi_k(\theta) = \cos(k\theta)$, it is well-known that
$$\hat{\psi}_k(u) = \frac{1}{2}(\delta(u-k) + \delta(u+k)).$$
- This facilitates the computation of $\tilde{A}_k$ avoiding the calculation of the integral. Thus, the minimum of R_{γ_n}

$$\tilde{A}_k = \frac{1}{2n}\left(\left(\frac{1}{1+\gamma_n(-k)^2\hat{\mathcal{K}}(-k)\hat{\mathcal{K}}(k)}\right)\sum_{j=1}^{n}\exp(ik\theta_j)\right.$$
$$\left. + \left(\frac{1}{1+\gamma_n k^2\hat{\mathcal{K}}(k)\hat{\mathcal{K}}(-k)}\right)\sum_{j=1}^{n}\exp(-ik\theta_j)\right)$$
$$= \frac{1}{1+\gamma_n k^2\hat{\mathcal{K}}(k)\hat{\mathcal{K}}(-k)}\frac{1}{n}\sum_{j=1}^{n}\cos(k\theta_j) = \frac{1}{1+\gamma_n k^{2(p+1)}}\frac{1}{n}\sum_{j=1}^{n}\cos(k\theta_j),$$

where $\theta_j \in (0,\pi)$ are given samples of the unknown distribution. In the last step, $\hat{\mathcal{K}}(u) = (iu)^p$ is used.

The data-driven COS method

- Employ the solution of the regularization problem for density estimation in the COS framework.
- In both, the density is assumed to be in the form of a cosine series expansion.
- The minimum of the functional is in terms of the expansion coefficients.
- Take advantage of the COS machinery: pricing options, Greeks, etc.
- The samples must follow risk-neutral measure (Monte Carlo paths).

The data-driven COS method

- Key idea: $\tilde{A}_k$ approximates A_k.
- Risk neutral samples from an asset at time T, $S_1(t), S_2(t), \ldots, S_n(t)$.
- With a logarithmic transformation, we have

$$Y_j := \log\left(\frac{S_j(T)}{K}\right).$$

- The regularization solution is defined in $(0, \pi)$, by transformation

$$\theta_j = \pi\frac{Y_j - a}{b - a},$$

- The boundaries a and b are defined as

$$a := \min_{1\leq j\leq n}(Y_j), \quad b := \max_{1\leq j\leq n}(Y_j).$$

The data-driven COS method - European options

- The A_k coefficients are replaced by the data-driven $\tilde{A}_k$

$$A_k \approx \tilde{A}_k = \frac{\frac{1}{n}\sum_{j=1}^{n} \cos\left(k\pi \frac{Y_j - a}{b-a}\right)}{1 + \gamma_n k^{2(p+1)}}.$$

- The ddCOS pricing formula for European options

$$\begin{aligned}\tilde{v}(x,t) &= \mathrm{e}^{-r(T-t)} {\sum_{k=0}^{\infty}}' \frac{\frac{1}{n}\sum_{j=1}^{n} \cos\left(k\pi \frac{Y_j - a}{b-a}\right)}{1 + \gamma_n k^{2(p+1)}} \cdot V_k \\ &= \mathrm{e}^{-r(T-t)} {\sum_{k=0}^{\infty}}' \tilde{A}_k V_k.\end{aligned}$$

- As in the original COS method, we must truncate the infinite sum to a finite number of terms N

$$\tilde{v}(x,t) = \mathrm{e}^{-r(T-t)} {\sum_{k=0}^{N}}' \tilde{A}_k V_k,$$

The data-driven COS method - Greeks

- Data-driven expressions for the Δ and Γ sensitivities.
- Define the corresponding sine coefficients as

$$\tilde{B}_k := \frac{\frac{1}{n}\sum_{j=1}^{n} \sin\left(k\pi \frac{Y_j-a}{b-a}\right)}{1+\gamma_n k^{2(p+1)}}.$$

- Taking derivatives of the ddCOS pricing formulat w.r.t the samples, Y_j, the data-driven Greeks, $\tilde{\Delta}$ and $\tilde{\Gamma}$, can be obtained by

$$\tilde{\Delta} = \mathrm{e}^{-r(T-t)} \sum_{k=0}^{N}{}' \tilde{B}_k \cdot \left(-\frac{k\pi}{b-a}\right) \cdot \frac{V_k}{S(0)},$$

$$\tilde{\Gamma} = \mathrm{e}^{-r(T-t)} \sum_{k=0}^{N}{}' \left(\tilde{B}_k \cdot \frac{k\pi}{b-a} - \tilde{A}_k \cdot \left(\frac{k\pi}{b-a}\right)^2\right) \cdot \frac{V_k}{S^2(0)}.$$

The data-driven COS method - Variance reduction

- Here, we show how to apply *antithetic variates (AV)* to our method.
- Since the samples must be i.i.d., an immediate application of AV is not possible.
- Assume antithetic samples, Y_i', that can be computed without extra computational effort, a new estimator is defined as

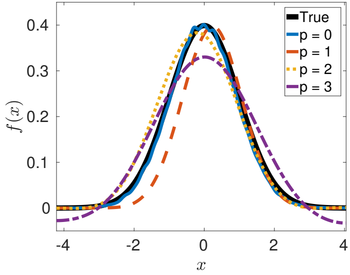
$$\bar{A}_k := \frac{1}{2}\left(\tilde{A}_k + \tilde{A}_k'\right),$$

where $\tilde{A}_k'$ are "antithetic coefficients", obtained from Y_i'.
- It can be proved that the use of $\bar{A}_k$ results in a variance reduction.
- Additional information to reduce the variance. For example, the martingale property

$$\begin{aligned} S(T) &= S(T) - \frac{1}{n}\sum_{j=1}^{n} S_j(T) + \mathbb{E}[S(T)], \\ &= S(T) - \frac{1}{n}\sum_{j=1}^{n} S_j(T) + S(0)\exp(rT). \end{aligned}$$

Choice of parameters in ddCOS method

- The choice of optimal values of γ_n and p.
- There is no rule or procedure to obtain an optimal p.
- As a rule of thumb, $p = 0$ seems to be the most appropriate value.



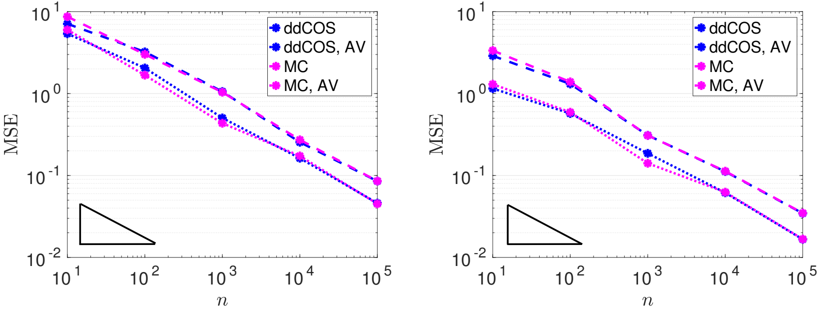
- For the regularization parameter γ_n, a rule that ensures asymptotic convergence

$$\gamma_n = \frac{\log \log n}{n}.$$

Applications of the ddCOS method

- Pricing options (no better than Monte Carlo).
- Sensitivities or Greeks.
- Models without analytic characteristic function. SABR model.
- Risk measures: VaR and Expected shortfall.
- Combinations.

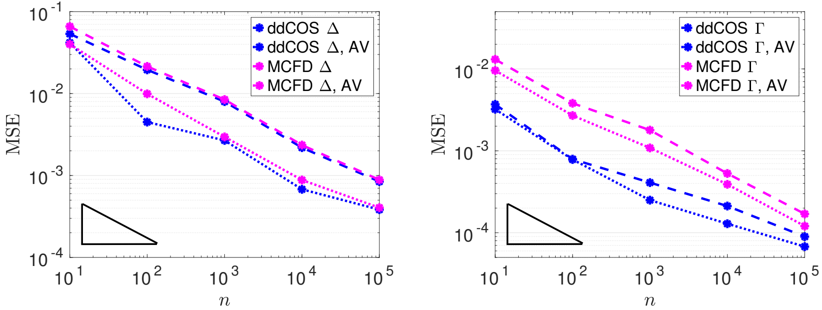
Applications of the ddCOS - Option pricing



(a) Call: Strike $K = 100$.   (b) Put: Strike $K = 100$.

Figure: Convergence in prices of the ddCOS method: Antithetic Variates (AV); GBM, $S(0) = 100$, $r = 0.1$, $\sigma = 0.3$ and $T = 2$.

Applications of the ddCOS - Greeks estimation



(a) Δ (Call): Strike $K = 100$.

(b) Γ: Strike $K = 100$.

Figure: Convergence in Greeks of the ddCOS method: Antithetic Variates (AV); GBM, $S(0) = 100$, $r = 0.1$, $\sigma = 0.3$ and $T = 2$.

Applications of the ddCOS - Greeks estimation

K (% of $S(0)$)	80%	90%	100%	110%	120%
0.1			Δ		
Ref.	0.8868	0.8243	0.7529	0.6768	0.6002
ddCOS	0.8867	0.8240	0.7528	0.6769	0.6002
RE	1.1012×10^{-4}				
MCFD	0.8876	0.8247	0.7534	0.6773	0.6006
RE	7.5168×10^{-4}				
			Γ		
Ref.	0.0045	0.0061	0.0074	0.0085	0.0091
ddCOS	0.0045	0.0062	0.0075	0.0084	0.0090
RE	8.5423×10^{-3}				
MCFD	0.0045	0.0059	0.0071	0.0079	0.0083
RE	4.9554×10^{-2}				

Table: GBM call option Greeks: $S(0) = 100$, $r = 0.1$, $\sigma = 0.3$ and $T = 2$.

Applications of the ddCOS - Greeks estimation

K (% of $S(0)$)	80%	90%	100%	110%	120%
			Δ		
Ref.	0.8385	0.8114	0.7847	0.7584	0.7328
ddCOS	0.8383	0.8113	0.7846	0.7585	0.7333
RE	2.7155×10^{-4}				
MCFD	0.8387	0.8118	0.7850	0.7586	0.7330
RE	3.1265×10^{-4}				
			Γ		
Ref.	0.0022	0.0024	0.0027	0.0029	0.0030
ddCOS	0.0022	0.0024	0.0027	0.0029	0.0030
RE	8.2711×10^{-3}				
MCFD	0.0023	0.0026	0.0028	0.0031	0.0033
RE	6.118×10^{-2}				

Table: Merton jump-diffusion call option Greeks: $S(0) = 100$, $r = 0.1$, $\sigma = 0.3$, $\mu_j = -0.2$, $\sigma_j = 0.2$ and $\lambda = 8$ and $T = 2$.

Applications of the ddCOS - Greeks estimation

K (% of $S(0)$)	80%	90%	100%	110%	120%
			Δ		
Ref.	0.9914	0.9284	0.5371	0.0720	0.0058
ddCOS	0.9916	0.9282	0.5363	0.0732	0.0058
RE	5.2775×10^{-3}				
MCFD	0.9911	0.9279	0.5368	0.0737	0.0058
RE	5.5039×10^{-3}				

Table: Call option Greek Δ under the SABR model: $S(0) = 100$, $r = 0$, $\sigma_0 = 0.3$, $\alpha = 0.4$, $\beta = 0.6$, $\rho = -0.25$ and $T = 2$.

Applications of the ddCOS - Greeks estimation

K (% of $S(0)$)	80%	90%	100%	110%	120%
			Δ		
Ref.	0.8384	0.7728	0.6931	0.6027	0.5086
ddCOS	0.8364	0.7703	0.6902	0.6006	0.5084
RE	2.7855×10^{-3}				
Hagan	0.8577	0.7955	0.7170	0.6249	0.5265
RE	3.1751×10^{-2}				

Table: Δ under SABR model. Setting: Call, $S(0) = 0.04$, $r = 0.0$, $\sigma_0 = 0.4$, $\alpha = 0.8$, $\beta = 1.0$, $\rho = -0.5$ and $T = 2$.

Applications of the ddCOS - Risk measures

- In the context of the Delta-Gamma approach (COS in [OGO14]).
- The change in a portfolio value is defined:

$$L := -\Delta V = V(S,t) - V(S+\Delta S, t+\Delta t).$$

- The formal definition of the VaR reads

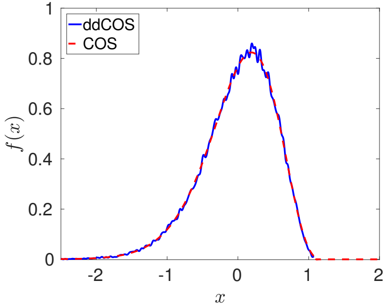
$$\mathbb{P}(\Delta V < \mathrm{VaR}(q)) = 1 - F_L(\mathrm{VaR}(q)) = q,$$

  with q a predefined confidence level.
- Given the VaR, the ES measure is computed as

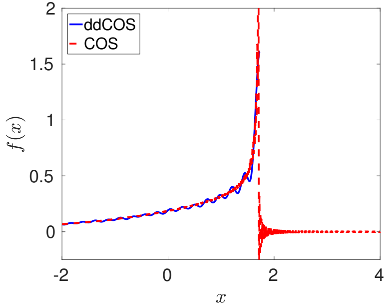
$$\mathrm{ES} := \mathbb{E}[\Delta V | \Delta V > \mathrm{VaR}(q)].$$

- Two portfolios with the same composition: one European call and half a European put on the same asset, maturity 60 days and $K = 101$.
- Different time horizons: 1 day (Portfolio 1) and 10 days (Portfolio 2). The asset follows a GBM with $S(0) = 100$, $r = 0.1$ and $\sigma = 0.3$.

Applications of the ddCOS - Risk measures



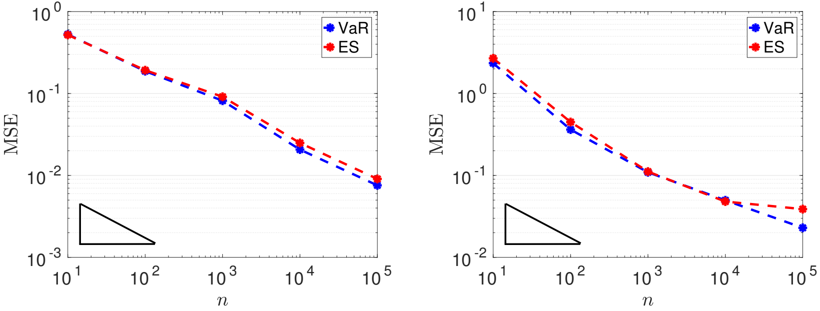
(a) Density Portfolio 1.



(b) Density Portfolio 2.

Figure: Recovered densities of L: ddCOS vs. COS.

Applications of the ddCOS - Risk measures



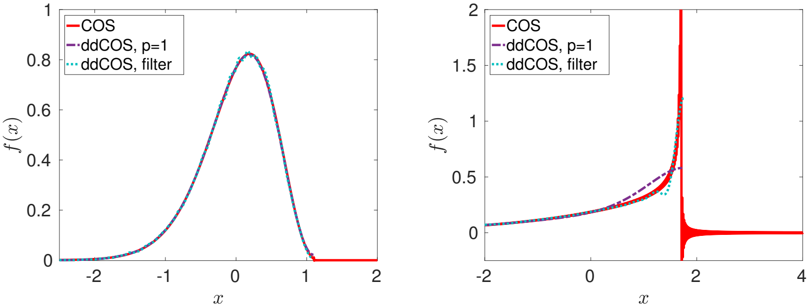
(a) Portfolio 1: $q = 99\%$.

(b) Portfolio 2: $q = 90\%$.

Figure: VaR and ES convergence in n.

Applications of the ddCOS - Risk measures

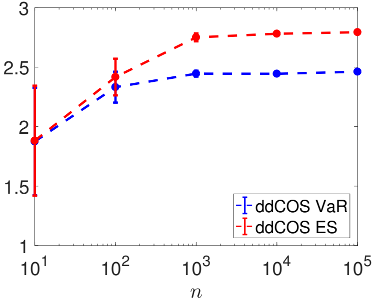
- The oscillations can be removed.
- Two options: smoothing parameter or filters [RVO14].



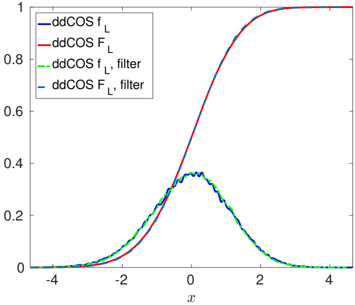
(a) Density Portfolio 1.

(b) Density Portfolio 2.

Figure: Smoothed densities of L.



(a) VaR and ES: $q = 99\%$.



(b) F_L and f_L.

Figure: Delta-Gamma approach under the SABR model. Setting: $S(0) = 100$, $K = 100$, $r = 0.0$, $\sigma_0 = 0.4$, $\alpha = 0.8$, $\beta = 1.0$, $\rho = -0.5$, $T = 2$, $q = 99\%$ and $\Delta t = 1/365$.

Applications of the ddCOS - Risk measures and SABR

q	10%	30%	50%	70%	90%
VaR	-1.4742	-0.5917	-0.0022	0.5789	1.3862
ES	0.1972	0.5345	0.8644	1.2517	1.8744

Table: VaR and ES under SABR model. Setting: $S(0) = 100$, $K = 100$, $r = 0.0$, $\sigma_0 = 0.4$, $\alpha = 0.8$, $\beta = 1.0$, $\rho = -0.5$, $T = 2$, and $\Delta t = 1/365$.

Conclusions

- The ddCOS method extends the COS method applicability to cases when only data samples of the underlying are available.
- The method exploits a closed-form solution, in terms of Fourier cosine expansions, of a regularization problem.
- It allows to develop a data-driven method which can be employed for option pricing and risk management.
- The ddCOS method particularly results in an efficient method for the Δ and Γ sensitivities computation, based solely on the samples.
- It can be employed within the Delta-Gamma approximation for calculating risk measures.

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Suggestions, comments & questions

Thank you for your attention

Choice of γ_n

- γ_n impacts the efficiency of the ddCOS method: it is related to the number of samples, n, and number of terms, N.
- For the regularization parameter γ_n, a rule that ensures asymptotic convergence

$$\gamma_n = \frac{\log \log n}{n}.$$

- In practical situations: not optimal.
- Exploit the relation between the empirical and real (unknown) CDFs.

Choice of γ_n

- This relation can be modeled by *statistical laws* or *statistics*: Kolmogorov-Smirnov, Anderson-Darling, Smirnov-Cramér–von Mises.
- Preferable: a measure of the distance between the $F_n(x)$ and $F(x)$ follows a known distribution.
- We have chosen Smirnov-Cramér–von Mises(SCvM):

$$\omega^2 = n \int_{\mathbb{R}} (F(x) - F_n(x))^2 \, \mathrm{d}F(x).$$

- Assume we have an approximation, F_{γ_n} (which depends on γ_n).
- An *almost* optimal γ_n is computed by solving the equation

$$\sum_{i=1}^{n} \left(F_{\gamma_n}(\bar{X}_i) - \frac{i-0.5}{n} \right)^2 = m_S - \frac{i}{12n},$$

where $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n$ is the ordered array of samples $X_1, X_2, \ldots, X_n$ and m_S the mean of the ω^2.

Influence of γ_n

- To assess the impact of γ_n: *Mean integrated Squared Error* (MiSE):

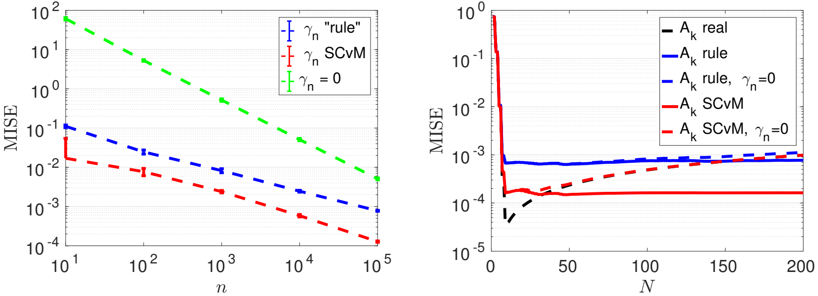
$$\mathbb{E}\left[\|f_n - f\|_2^2\right] = \mathbb{E}\left[\int_{\mathbb{R}} (f_n(x) - f(x))^2 \,\mathrm{d}x\right].$$

- A formula for the MiSE formula is derived in our context:

$$\text{MISE} = \frac{1}{n}\sum_{k=1}^{N} \frac{1}{\left(1+\gamma_n k^{2(p+1)}\right)^2}\left(\frac{1}{2}+\frac{1}{2}A_{2k}-A_k^2\right) + \sum_{k=N+1}^{\infty} A_k^2.$$

- Two main aspects influenced γ_n: accuracy in n and stability in N.
- The quality of the approximated density can be also affected.

Influence of γ_n



(a) Convergence in terms of n

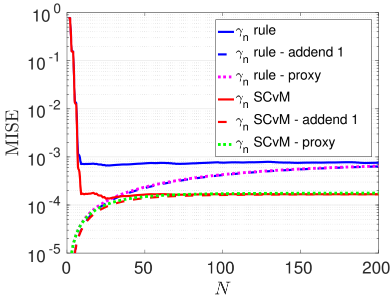
(b) Convergence in terms of N

Figure: Influence of γ_n: .

Optimal number of terms N

- Try to find a *minimum optimal* value of N.
- N considerably affects the performance.
- We wish to avoid the computation of any $\hat{A}_k$.
- We define a proxy for the MiSE and follow:

$$\text{MiSE} \approx \frac{1}{n} \sum_{k=1}^{N} \frac{\frac{1}{2}}{\left(1 + \gamma_n k^{2(p+1)}\right)^2}.$$



Optimal number of terms N

Data: n, γ_n

$N_{min} = 5$

$N_{max} = \infty$

$\epsilon = \frac{1}{\sqrt{n}}$

$\text{MiSE}_{prev} = \infty$

for $N = N_{min} : N_{max}$ **do**

$\quad \text{MiSE}_N = \frac{1}{n}\sum_{k=1}^{N} \frac{\frac{1}{2}}{\left(1+\gamma_n k^{2(p+1)}\right)^2}$

$\quad \epsilon_N = \frac{|\text{MiSE}_N - \text{MiSE}_{prev}|}{|\text{MiSE}_N|}$

$\quad$ **if** $\epsilon_N > \epsilon$ **then**

$\quad\quad N_{op} = N$

$\quad$ **else**

$\quad\quad$ Break

$\quad \text{MiSE}_{prev} = \text{MiSE}_N$

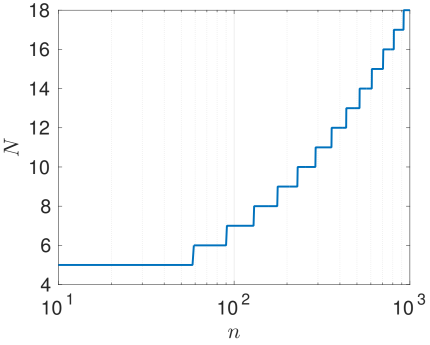


Figure: Almost optimal N.