Continuous Time Markov Chain approximation of the Heston model

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Motivation

- The Heston model is a widely utilized stochastic volatility (SV) models in the option pricing literature as well as in practice.
- For a fixed time horizon, the characteristic function (ChF) is known in closed-form.
- Then, European option pricing is efficiently accomplished with any standard Fourier method.
- Enabling a fast calibration of the Heston model parameters to match observed volatility surfaces, as required in practice.
- However, after calibration there is still great difficulty in pricing exotic contracts under the Heston model.
- To price contracts such as Asian options and variance swaps, Monte Carlo (MC) methods are the traditional surrogates in these cases.
- Unfortunately, MC suffers from a number of well known deficiencies, and complicated simulation schemes are often required to overcome the boundary effects that accompany models such as Heston.
The practical objective of this work is to formalize a model which reproduces vanilla market quotes, but is at the same time amenable to complex derivative pricing in a manner that is consistent with the calibrated model.

We propose a model and framework based on the Heston model. We call this the CTMC-Heston model, as it uses a finite state Continuous Time Markov Chain (CTMC) approximation to the variance process.

The new formulation enables a closed-form solution for the ChF of the underlying (log-)returns, which allows the use of Fourier inversion techniques to efficiently price exotics.

We provide numerical studies which demonstrate convergence to Heston’s model as the state space is refined. A detailed theoretical analysis of the method will follow.
Outline

1. From Heston model to CTMC-Heston model
2. Calibration of the CTMC-Heston model
3. Application: Pricing Exotic options under CTMC-Heston model
4. Conclusions
The Heston stochastic volatility model,

\[
\frac{dS_t}{S_t} = (r - q)dt + \sqrt{v_t}dW^1_t,
\]
\[
dv_t = \eta(\theta - v_t)dt + \sigma_v \sqrt{v_t}dW^2_t,
\]

where \(dW^1_t\) and \(dW^2_t\) are correlated Brownian motions, i.e. \(dW^1_t dW^2_t = \rho dt\), with \(\rho \in (-1, 1)\).

- The stochastic volatility (or variance), \(v_t\), is driven by a CIR process, having a mean reversion component.
- Value \(v_0\) is the initial volatility, \(\eta\) controls the mean reversion speed while \(\theta\) is the long-term volatility and \(\sigma_v\) corresponds to the volatility of the variance process \(v_t\), also known as vol-vol (volatility-of-volatility).
- The model parameters are therefore \(\Theta = \{v_0, \eta, \theta, \sigma_v, \rho\}\).
The Heston’s model solution can be re-expressed in the form

\[
\log \left( \frac{S_t}{S_0} \right) = \frac{\rho}{\sigma_v} (\nu_t - \nu_0) + (r - q) t - \frac{1}{2} \int_0^t \nu_s ds \\
- \frac{\rho}{\sigma_v} \int_0^t \eta(\theta - \nu_s) ds + \sqrt{1 - \rho^2} \int_0^t \sqrt{\nu_s} \, dW_s^*,
\]

where \( W^1_t := \rho W^2_t + \sqrt{1 - \rho^2} W^*_t \) and \( W^*_t \) is independent from \( W^2_t \).

Rearranging, we introduce the auxiliary process \((\tilde{X}_t)_{t \geq 0}\),

\[
\tilde{X}_t := \log \left( \frac{S_t}{S_0} \right) - \frac{\rho}{\sigma_v} (\nu_t - \nu_0) \\
= \left( r - q - \frac{\rho \eta \theta}{\sigma_v} \right) t + \left( \frac{\rho \eta}{\sigma_v} - \frac{1}{2} \right) \int_0^t \nu_s ds + \sqrt{1 - \rho^2} \int_0^t \sqrt{\nu_s} \, dW^*_s.
\]

We thus have the following uncoupled two-factor representation,

\[
d\tilde{X}_t = \left[ \left( \frac{\rho \eta}{\sigma_v} - \frac{1}{2} \right) \nu_t + \tilde{\omega} \right] dt + \sqrt{(1 - \rho^2) \nu_t} \, dW^*_t, \\
d\nu_t = \mu(\nu_t) dt + \sigma(\nu_t) dW^2_t,
\]

where \( \tilde{\omega} := (r - q - \frac{\rho \eta \theta}{\sigma_v}), \mu(\nu_t) := \eta(\theta - \nu_t) \) and \( \sigma(\nu_t) := \sigma_v \sqrt{\nu_t} \).
CTMC-Heston model

- Given a state-space $\mathbf{v} := \{v_1, \ldots, v_{m_0}\}$, and a CTMC $\{\alpha(t), t \geq 0\}$ transitioning between the indexes $\{1, \ldots, m_0\}$ according to
  $$Q\{\alpha(t + \Delta t) = j | \alpha(t) = k\} = \delta_{jk} + q_{jk} \Delta t + o(\Delta t).$$

- The set of transition rates $q_{jk}$ form the generator matrix $Q_{m_0 \times m_0}$, chosen so that $(v_{\alpha(t)})_{t \geq 0}$ are locally consistent with $(v_t)_{t \geq 0}$.

- Given $(v_{\alpha(t)})_{t \geq 0}$, $\tilde{X}_t$ is approximated by a Regime Switching (RS) diffusion,
  $$\tilde{X}_t^\alpha = \bar{\omega} t + \int_0^t \left( \frac{\rho \eta \sigma_v}{\sigma_v} - \frac{1}{2} \right) v_{\alpha(s)} ds + \sqrt{1 - \rho^2} \int_0^t \sqrt{v_{\alpha(s)}} dW^*(s)$$
  $$= \int_0^t \zeta_{\alpha(s)} ds + \int_0^t \beta_{\alpha(s)} dW^*(s),$$

where for $\alpha(s) \in \{1, \ldots, m_0\}$,

$$\zeta_{\alpha(s)} := \left( r - q - \frac{\rho \eta \theta}{\sigma_v} \right) + \left( \frac{\rho \eta}{\sigma_v} - \frac{1}{2} \right) v_{\alpha(s)}, \quad \beta_{\alpha(s)} := \sqrt{(1 - \rho^2)v_{\alpha(s)}}.$$
Main advantage: the new formulation enables a closed-form expression for the conditional ChF. Given $\Delta t > 0$, $\forall j = 1, \ldots, m_0$,

$$
\tilde{\phi}^j_{\tilde{X}_{\Delta t}^\alpha}(\xi) := \mathbb{E}[e^{i\xi \tilde{X}_{\Delta t}^\alpha} | \alpha(0 \leq s \leq \Delta t) = j] = \mathbb{E} \left[ \exp \left( i\xi (\zeta_j \Delta t + \beta_j W^*(\Delta t)) \right) \right] := \exp(\psi_j(\xi)\Delta t),
$$

where $\psi_j(\xi) = i\zeta_j \xi - \frac{1}{2} \xi^2 \beta_j^2$, $j = 1, \ldots, m_0$ is its Lévy symbol.

The process $\tilde{X}_{\Delta t}^\alpha$ is completely characterized by the set $\{\psi_j(\xi)\}_{j=1}^{m_0}$, together with the generator $Q$.

The ChF of $\tilde{X}_{\Delta t}^\alpha$, $\Delta t \geq 0$, conditioned on the initial state $\alpha(0) = j_0$,

$$
\mathbb{E} \left[ e^{i\tilde{X}_{\Delta t}^\alpha\xi} | \alpha(0) = j_0 \right] = 1'M(\xi; \Delta t)e_{j_0}, \quad j_0 \in \{1, \ldots, m_0\}
$$

where we define the matrix exponential

$$
M(\xi; \Delta t) := \exp \left( \Delta t \left( Q' + \text{diag}(\psi_1(\xi), \ldots, \psi_{m_0}(\xi)) \right) \right),
$$

and $1 \in \mathbb{R}^{m_0}$ represents a column vector of ones, and $e_j \in \mathbb{R}^{m_0}$ a unit column vector with a one in position $j$. 

\( \tilde{X}_{\Delta t}^{\alpha} \) induces the following CTMC-Heston model for the underlying \( S_{\Delta t} \), namely

\[
S_{\Delta t}^{\alpha} = S_0 \exp \left( \tilde{X}_{\Delta t}^{\alpha} + \frac{\rho}{\sigma_v} (v_{\alpha(\Delta t)} - v_{\alpha(0)}) \right).
\]

The conditional ChF of the log-increment

\[
R_{\Delta t}^{\alpha} := \log(S_{\Delta t}^{\alpha}/S_0) = \tilde{X}_{\Delta t}^{\alpha} + \frac{\rho}{\sigma_v} (v_{\alpha(\Delta t)} - v_{\alpha(0)}),
\]

is recovered in closed-form as

\[
\mathbb{E}[e^{iR_{\Delta t}^{\alpha}\xi}|\alpha(0) = j, \alpha(\Delta t) = k] = \mathcal{M}_{k,j}(\xi; \Delta t) \cdot \exp \left( i\xi \frac{\rho}{\sigma_v} (v_k - v_j) \right) := \tilde{\mathcal{M}}_{k,j}(\xi; \Delta t).
\]

which follows from conditional independence.

We can view the CTMC-Heston model as both an approximation to Heston’s model, as well as a tractable model in its own right.
As a Fourier inversion method we employ *SWIFT*, which has several important advantages which make it well-suited for calibration:

- **Error control.** It is probably the most relevant property within an optimization problem. Thanks to the use of Shannon wavelets, SWIFT establishes a bound in the error given any scale $m$ of approximation.

- **Robustness.** SWIFT provides mechanisms to determine all the free parameters in the approximation made based on the scale $m$ which, as mentioned in the previous point, determines the committed error.

- **Performance efficiency.** As other Fourier inversion techniques, SWIFT is an extremely fast algorithm, allowing FFT, vectorized operations or even parallel computing features.

- **Accuracy.** Although an error bound is provided, SWIFT has demonstrated a very high precision in most situations, far below the predicted error bound and, at least, comparable with the state-of-the-art methodologies.

The properties mentioned above ensure high quality estimations in the calibration process, reducing the chances of any possible malfunctioning or divergence in the optimization procedure.
Our goal is to form a model which parsimoniously resembles Heston. One of the key aspects in designing the CTMC-Heston model is a specification for the variance state-space (grid). Several conceptually different approaches available in the literature.
CTMC: numerical study

- Data sets: two representative scenarios.

<table>
<thead>
<tr>
<th>scenario</th>
<th>$v_0$</th>
<th>$\eta$</th>
<th>$\theta$</th>
<th>$\sigma_v$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set I regular market</td>
<td>0.03</td>
<td>3.0</td>
<td>0.04</td>
<td>0.25</td>
<td>-0.7</td>
</tr>
<tr>
<td>Set II stressed market</td>
<td>0.4</td>
<td>3.0</td>
<td>0.4</td>
<td>0.5</td>
<td>-0.1</td>
</tr>
</tbody>
</table>

- Convergence in $m_0$. 

![Graph](attachment:image.png)

(a) Set I
(b) Set II

Figure: Market parameters: put option, $S_0 = 100$, $K = 100$, $r = 0.05$ and $T = 1$. 

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Influence of the model parameters

**Figure:** Set I: put option, \( S_0 = 100, \ K = 100, \ r = 0.05 \) and \( T = 1 \).
Influence of the model parameters

Figure: Set II: put option, $S_0 = 100$, $K = 100$, $r = 0.05$ and $T = 1$. 
Calibration with real data (Microsoft, January 2019)

Figure: Microsoft calibration curves for varying $m_0$. Market parameters: call options, $S_0 = 105.36$, $K = \{65, 70, \ldots, 150, 155\}$, $r = 0.0246$ and $T = 0.4986$.

Heston parameters: $v_0 = 0.0906$, $\eta = 0.8549$, $\theta = 0.1379$, $\sigma_v = 0.9976$, $\rho = -0.6187$. 

(a) Uniform.
(b) Mijatovic-Pistorius
(c) Tavella-Randall
(d) Lo-Skindilias
Calibration with real data (Google, January 2019)

(a) Uniform.

(b) Mijatovic-Pistorius

(c) Tavella-Randall

(d) Lo-Skindilias

Figure: Google implied volatility: call options, $S_0 = 1080.66$, $K = \{880, 890, \ldots, 1390, 1395\}$, $r = 0.0249$ and $T = 0.9972$; $\nu_0 = 0.1482$, $\eta = 0.7752$, $\theta = 0.0722$, $\sigma_v = 0.9278$, $\rho = -0.5444$. 

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CTMC-Heston model

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Interesting lessons

- All the approaches provide numerical convergence in $m_0$.
- The error decays very fast at the beginning, with smaller $m_0$, and smoothens for bigger $m_0$, suggesting a damping effect.
- The grid distribution proposed by Lo and Skindilias provides, in general, poorer estimations.
- Although the uniform approach performs surprisingly well in the test with synthetic parameters, the real calibration experiment shows a pretty inaccurate estimations for options far from at-the-money strike.
- The schemes by Mijatovic-Pistorius and Tavella-Randall perform similarly. It is worth noting that the first explodes when the initial and long-term volatilities differ greatly one from the other. The second happens to be the most robust and precise choice in general.
- By focusing on the correlation parameter, $\rho$, in the second test, we observe that the error tends to be minimum close to the no-correlation point ($\rho = 0$), and it degrades when $\rho$ ventures far form zero.
Once calibrated, a model is commonly employed to price more involved products (early-exercise, path-dependent, etc.).

Many exotic products can be defined in terms of a generic recursion.

Consider $N + 1$ monitoring dates, $0 = t_0 < t_1 < \cdots < t_N = T$. We define the log returns $R_n$ by

$$R_n := \log \left( \frac{S_n}{S_{n-1}} \right), \quad S_n := S(t_n), \quad n = 1, \ldots, N.$$ 

The contracts of interest satisfy a very general sequence of equations

$$Y_1 := w_N \cdot h(R_N) + \varrho_N$$
$$Y_n := w_{N-(n-1)} \cdot h(R_{N-(n-1)}) + g(Y_{n-1}) + \varrho_{N-(n-1)}, \quad n = 2, \ldots, N,$$

where $h, g$ are continuous functions, $\{w_n\}_{n=1}^N$ is a set of weights, and $\{\varrho_n\}_{n=1}^N$ is a set of shift parameters. Includes contracts of the form

$$G \left( \sum_{n=1}^{N} w_n \cdot h(R_n); \Theta \right).$$
Prominent examples of contracts which fall within this framework.

- **Realized variance swaps and options:**

  \[ A_N = \frac{1}{T} \sum_{n=1}^{N} (R_n)^2 \quad \text{and} \quad A_N = \frac{1}{T} \sum_{n=1}^{N} (\exp(R_n) - 1)^2, \]

  with \( G(A_N) := A_N - K \) (swap), and \( G(A_N) := (A_N - K)^+ \) (call).

- **Cliquets:** with local (global) floor and cap \( F, G \) \((F_g, G_g)\),

  \[ A_N = \sum_{n=1}^{N} \max (F, \min (C, \exp(R_n) - 1)), \]

  with \( G(A_N) = K \cdot \min (C_g, \max (F_g, A_N)) \).

- **Arithmetic (weighted) Asian Options:**

  \[ A_N := \frac{1}{N+1} \sum_{n=0}^{N} w_n S_n \]

  \[ = \frac{S_0}{N+1} \left( w_0 + e^{R_1} (w_1 + e^{R_2} (\cdots e^{R_{N-1}} (w_{N-1} + w_N e^{R_N}))) \right), \]

  where \( G(A_N) := (A_N - K)^+ \) for a call option.
Numerical experiments with exotic options

- We will present some experiments aiming to numerically validate the introduced CTMC-Heston model.
- We will consider several exotic contracts: realized variance swaps, realized variance options and Asian options.
- The recursive definition above allows efficient Fourier methods (SWIFT).
- The realized variance swaps are chosen for comparative purposes, since an exact solution for the Heston model is available.
- That is not the case for the other two products, which often require the use of MC methods.
- Computer system CPU Intel Core i7-4720HQ 2.6GHz, 16GB RAM and Matlab R2017b.
- Based on the calibration tests, Tavella-Randall scheme is used.
- MC setting: QE scheme with $10^6$ paths and 360 time steps.
Convergence in $m_0$

**Figure:** Variance Swaps: $r = 0.05$ and $T = 1$. Heston parameters: Set I (regular market). Grid Design: Tavella-Randall.

(a) $\rho = -0.1$.

(b) $\rho = -0.7$. 
Convergence in $m_0$

(a) $\rho = -0.1$.

(b) $\rho = -0.7$.

Figure: Variance Swaps: $r = 0.05$ and $T = 1$. Heston parameters: Set II (stressed market). Grid Design: Tavella-Randall.
Realized variance swaps (Set I)

Table: Variance Swaps: $m_0 = 40$, Set I, $T = 1$ $r = 0.05$. 

<table>
<thead>
<tr>
<th>$N$</th>
<th>Ref.</th>
<th>SWIFT</th>
<th>MC</th>
<th>$R_{ESWIFT}$</th>
<th>$R_{EMC}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.0371205474</td>
<td>0.0371205820</td>
<td>0.0371242281</td>
<td>$9.32 \times 10^{-7}$</td>
<td>$9.91 \times 10^{-5}$</td>
</tr>
<tr>
<td>12</td>
<td>0.0369570905</td>
<td>0.0369571055</td>
<td>0.0369627242</td>
<td>$4.06 \times 10^{-7}$</td>
<td>$1.52 \times 10^{-4}$</td>
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<tr>
<td>50</td>
<td>0.0368631686</td>
<td>0.0368630034</td>
<td>0.0368736021</td>
<td>$4.48 \times 10^{-6}$</td>
<td>$2.83 \times 10^{-4}$</td>
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<tr>
<td>180</td>
<td>0.0368411536</td>
<td>0.0368414484</td>
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<td>$8.00 \times 10^{-6}$</td>
<td>$1.46 \times 10^{-4}$</td>
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<tr>
<td>360</td>
<td>0.0368368930</td>
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<td>0.0368466261</td>
<td>$7.23 \times 10^{-5}$</td>
<td>$2.64 \times 10^{-4}$</td>
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</table>

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<tr>
<th>$N$</th>
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<th>MC</th>
<th>$R_{ESWIFT}$</th>
<th>$R_{EMC}$</th>
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<tr>
<td>5</td>
<td>0.0375737983</td>
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<td>180</td>
<td>0.0368564120</td>
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<td>$1.35 \times 10^{-4}$</td>
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<tr>
<td>360</td>
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<td>$1.18 \times 10^{-4}$</td>
<td>$2.09 \times 10^{-4}$</td>
</tr>
</tbody>
</table>
Realized variance swaps (Set II)

\[ \rho = -0.1 \]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
N & \text{Ref.} & \text{SWIFT} & \text{MC} & \text{RE}_{\text{SWIFT}} & \text{RE}_{\text{MC}} \\
\hline
5  & 0.4067078727 & 0.4067086532 & 0.4065423061 & 1.91 \times 10^{-6} & 4.07 \times 10^{-4} \\
12 & 0.4029056015 & 0.4029060040 & 0.4027957415 & 9.99 \times 10^{-7} & 2.72 \times 10^{-4} \\
50 & 0.4007139003 & 0.4007139406 & 0.4008416719 & 1.00 \times 10^{-7} & 3.18 \times 10^{-4} \\
180 & 0.4001994199 & 0.4001993773 & 0.4002238554 & 1.06 \times 10^{-7} & 6.10 \times 10^{-5} \\
360 & 0.4000998185 & 0.4000997601 & 0.4000181976 & 1.46 \times 10^{-7} & 2.04 \times 10^{-4} \\
\hline
\end{array}
\]

\[ \rho = -0.7 \]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
N & \text{Ref.} & \text{SWIFT} & \text{MC} & \text{RE}_{\text{SWIFT}} & \text{RE}_{\text{MC}} \\
\hline
5  & 0.4166286485 & 0.4166310417 & 0.4167251660 & 5.74 \times 10^{-6} & 2.31 \times 10^{-4} \\
12 & 0.4075137267 & 0.4075104743 & 0.4073002992 & 7.98 \times 10^{-6} & 5.23 \times 10^{-4} \\
50 & 0.4018902561 & 0.4018867030 & 0.4019702179 & 8.84 \times 10^{-6} & 1.98 \times 10^{-4} \\
180 & 0.4005309091 & 0.4005272887 & 0.4006611714 & 9.03 \times 10^{-6} & 3.25 \times 10^{-4} \\
360 & 0.4002660232 & 0.4002623900 & 0.4001430561 & 9.07 \times 10^{-6} & 3.07 \times 10^{-4} \\
\hline
\end{array}
\]

**Table:** Variance Swaps: \( m_0 = 40 \), Set II, \( T = 1 \) \( r = 0.05 \).
Realized variance option

<table>
<thead>
<tr>
<th>$K$</th>
<th>Ref.(MC)</th>
<th>SWIFT</th>
<th>$RE$</th>
<th>$\rho = -0.1$</th>
<th>Ref.(MC)</th>
<th>SWIFT</th>
<th>$RE$</th>
<th>$\rho = -0.7$</th>
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<td>0.01701443</td>
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<td>0.01710650</td>
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<tr>
<td>0.03</td>
<td>0.01045427</td>
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<td>$1.09 \times 10^{-3}$</td>
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<tr>
<td>0.05</td>
<td>0.00351388</td>
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<td>$1.62 \times 10^{-3}$</td>
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</tr>
</tbody>
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Table: Variance Call Options: $m_0 = 40$, $T = 1$, $r = 0.05$, $N = 12$. Heston Set I.

<table>
<thead>
<tr>
<th>$K$</th>
<th>Ref.(MC)</th>
<th>SWIFT</th>
<th>$RE$</th>
<th>$\rho = -0.1$</th>
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<th>SWIFT</th>
<th>$RE$</th>
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<tr>
<td>0.4</td>
<td>0.07050097</td>
<td>0.07054838</td>
<td>$6.72 \times 10^{-4}$</td>
<td>0.07568155</td>
<td>0.07567259</td>
<td>$1.18 \times 10^{-4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.03826162</td>
<td>0.03836334</td>
<td>$2.65 \times 10^{-3}$</td>
<td>0.04341057</td>
<td>0.04352151</td>
<td>$2.55 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table: Variance Call Options: $m_0 = 40$, $T = 1$, $r = 0.05$, $N = 12$. Heston Set II.
### Arithmetic Asian option (Set I)

#### Table: Tavella-Randall, \( m_0 = 40, \) Set I, call, option, \( S_0 = 100, T = 1, r = 0.05 \)

<table>
<thead>
<tr>
<th>( K(% \text{of} S_0) )</th>
<th>( N = 12 )</th>
<th>( N = 50 )</th>
<th>( N = 250 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ref.(MC)</td>
<td>SWIFT</td>
<td>RE</td>
</tr>
<tr>
<td>80%</td>
<td>21.5285835237</td>
<td>21.5270366207</td>
<td>7.18 \times 10^{-5}</td>
</tr>
<tr>
<td>90%</td>
<td>12.5823808044</td>
<td>12.5896547750</td>
<td>5.78 \times 10^{-4}</td>
</tr>
<tr>
<td>100%</td>
<td>5.4002621022</td>
<td>5.4000546644</td>
<td>3.84 \times 10^{-5}</td>
</tr>
<tr>
<td>110%</td>
<td>1.3880527793</td>
<td>1.3906598970</td>
<td>1.87 \times 10^{-3}</td>
</tr>
<tr>
<td>120%</td>
<td>0.1736330491</td>
<td>0.1731034094</td>
<td>3.05 \times 10^{-3}</td>
</tr>
<tr>
<td></td>
<td>21.5386392371</td>
<td>21.5339280578</td>
<td>2.18 \times 10^{-4}</td>
</tr>
<tr>
<td>80%</td>
<td>12.6239658563</td>
<td>12.6182127196</td>
<td>4.55 \times 10^{-4}</td>
</tr>
<tr>
<td>90%</td>
<td>5.4504220302</td>
<td>5.4499634141</td>
<td>8.41 \times 10^{-5}</td>
</tr>
<tr>
<td>100%</td>
<td>1.4295579101</td>
<td>1.4275471949</td>
<td>1.40 \times 10^{-3}</td>
</tr>
<tr>
<td>110%</td>
<td>0.1824925012</td>
<td>0.1831473714</td>
<td>3.58 \times 10^{-3}</td>
</tr>
<tr>
<td></td>
<td>21.5266346261</td>
<td>21.5359313401</td>
<td>4.31 \times 10^{-4}</td>
</tr>
<tr>
<td>80%</td>
<td>12.6269859960</td>
<td>12.6261325064</td>
<td>6.75 \times 10^{-5}</td>
</tr>
<tr>
<td>90%</td>
<td>5.4534882341</td>
<td>5.4636939471</td>
<td>1.87 \times 10^{-3}</td>
</tr>
<tr>
<td>100%</td>
<td>1.4440819439</td>
<td>1.4378101025</td>
<td>4.34 \times 10^{-3}</td>
</tr>
<tr>
<td>110%</td>
<td>0.1875776074</td>
<td>0.1860298190</td>
<td>8.25 \times 10^{-3}</td>
</tr>
</tbody>
</table>
### Arithmetic Asian option (Set II)

#### $N = 12$

<table>
<thead>
<tr>
<th>$K$ (% of $S_0$)</th>
<th>Ref. (MC)</th>
<th>SWIFT</th>
<th>$RE$</th>
</tr>
</thead>
<tbody>
<tr>
<td>80%</td>
<td>25.5585678735</td>
<td>25.5988860602</td>
<td>$1.57 \times 10^{-3}$</td>
</tr>
<tr>
<td>90%</td>
<td>19.6670725943</td>
<td>19.6278689575</td>
<td>$1.99 \times 10^{-3}$</td>
</tr>
<tr>
<td>100%</td>
<td>14.8962382700</td>
<td>14.8552716759</td>
<td>$2.75 \times 10^{-3}$</td>
</tr>
<tr>
<td>110%</td>
<td>11.1517895745</td>
<td>11.1463503256</td>
<td>$4.87 \times 10^{-4}$</td>
</tr>
<tr>
<td>120%</td>
<td>8.3165299338</td>
<td>8.3212712111</td>
<td>$5.70 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

#### $N = 50$

<table>
<thead>
<tr>
<th>$K$ (% of $S_0$)</th>
<th>Ref. (MC)</th>
<th>SWIFT</th>
<th>$RE$</th>
</tr>
</thead>
<tbody>
<tr>
<td>80%</td>
<td>25.7824036750</td>
<td>25.7778794489</td>
<td>$1.75 \times 10^{-4}$</td>
</tr>
<tr>
<td>90%</td>
<td>19.8263899575</td>
<td>19.8272466858</td>
<td>$4.32 \times 10^{-5}$</td>
</tr>
<tr>
<td>100%</td>
<td>15.0530165896</td>
<td>15.0529723969</td>
<td>$2.93 \times 10^{-6}$</td>
</tr>
<tr>
<td>110%</td>
<td>11.3291439277</td>
<td>11.3270969069</td>
<td>$1.80 \times 10^{-4}$</td>
</tr>
<tr>
<td>120%</td>
<td>8.4614191560</td>
<td>8.4772028373</td>
<td>$1.86 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

#### $N = 250$

<table>
<thead>
<tr>
<th>$K$ (% of $S_0$)</th>
<th>Ref. (MC)</th>
<th>SWIFT</th>
<th>$RE$</th>
</tr>
</thead>
<tbody>
<tr>
<td>80%</td>
<td>25.8641465886</td>
<td>25.8255340288</td>
<td>$1.49 \times 10^{-3}$</td>
</tr>
<tr>
<td>90%</td>
<td>19.9219203435</td>
<td>19.8806560654</td>
<td>$2.07 \times 10^{-3}$</td>
</tr>
<tr>
<td>100%</td>
<td>15.1245760541</td>
<td>15.1064333350</td>
<td>$1.19 \times 10^{-3}$</td>
</tr>
<tr>
<td>110%</td>
<td>11.3793305624</td>
<td>11.3765266399</td>
<td>$2.46 \times 10^{-4}$</td>
</tr>
<tr>
<td>120%</td>
<td>8.5254366308</td>
<td>8.5203790223</td>
<td>$5.93 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

**Table:** Tavella-Randall, $m_0 = 40$, Set II, call, option, $S_0 = 100$, $T = 1$, $r = 0.05$. 

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Á. Leitao & J.L. Kirkby & L. Ortiz-Gracia

CTMC-Heston model

July 12, 2019 27 / 30
Conclusions

- This work provides a general, computationally efficient, and robust valuation framework under the CTMC-Heston model.
- This model approximation provides a parsimonious and faithful representation of the Heston model, and it is able to reproduce the same volatility smile structure with a modest number of states.
- We can efficiently price a large variety of contracts which are exceptionally difficult to handle under Heston’s model.
- The efficiency of the method is obtained by combining the CTMC approximation of the variance, with the SWIFT Fourier method.
- An extensive set of numerical experiments were provided, analyzing Asian options and discretely sampled realized variance derivatives.
- A detailed error analysis will follow (work in progress).
References

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Aleksandar Mijatović and Martijn Pistorius. 
Continuously monitored barrier options under Markov processes. 

Domingo Tavella and Curt Randall. 
Acknowledgements & Questions

Thanks to support from MDM-2014-0445

More: alvaroleitao.github.io

Thank you for your attention
Given a grid of points \( \mathbf{v} = \{v_1, v_2, \ldots, v_{m_0}\} \) with grid spacings \( h_i = v_{i+1} - v_i \), and assuming that \( v_{\alpha(t)} \) takes values on \( \mathbf{v} \), the elements \( q_{ij} \) of the generator \( Q \) for the CTMC approximation of the process \( v_t \) read

\[
q_{ij} = \begin{cases} 
\frac{\mu^-(v_i)}{h_{i-1}} + \frac{\sigma^2(v_i) - (h_{i-1}\mu^-(v_i) + h_i\mu^+(v_i))}{h_{i-1}(h_{i-1} + h_i)}, & \text{if } j = i - 1, \\
\frac{\mu^+(v_i)}{h_i} + \frac{\sigma^2(v_i) - (h_{i-1}\mu^-(v_i) + h_i\mu^+(v_i))}{h_i(h_{i-1} + h_i)}, & \text{if } j = i + 1, \\
-q_{i,i-1} - q_{i,i+1}, & \text{if } j = i, \\
0, & \text{otherwise,}
\end{cases}
\]

with the notation \( z^\pm = \max(\pm z, 0) \). Further, to guarantee a well-defined probability matrix, the following condition must be satisfied:

\[
\max_{1 \leq i < m_0} \left( h_i \right) \leq \min_{1 \leq i \leq m_0} \left( \frac{\sigma^2(v_i)}{|\mu(v_i)|} \right).
\]