The BENCHOP project

The BENCHmarking project in Option Pricing

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The BENCHOP project

- The purpose and aim of BENCHOP is to provide sets of benchmark problems.
- Facilitating comparison and evaluation of different methods.
- Expecting that future papers in the financial field will compare method performances with the methods in BENCHOP.
- Contributing to a more uniform comparison and understanding of different methods' pros and cons.
- Results published in a journal articles.
- This is the second edition. The results of the first edition can be found in [vSHL+15].

Aspects 2nd edition

- Implementation should be in Matlab.
- Preferable, use of high-performance features: parallel computing toolbox.
  - parfor.
  - GPU array.
- Two categories:
  - Basket options.
  - Stochastic and local volatility.
- Benchmark: Error (accuracy) in the solution as a function of CPU (GPU) time.

Basket options - Problem formulation

- Underlying prices modelled by a multidimensional Merton model:

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = (r - \lambda\kappa_i)\mathrm{d}t + \mathrm{d}B_i(t) + \left(\mathrm{e}^{J_i(t)} - 1\right)\mathrm{d}P(t).$$

- $\mathrm{d}B_i(t), \quad i = 1, \ldots, d$ is a multidimensional Brownian motion with covariance matrix $\Sigma_{ij}^B = \sigma_i^B(S_i, t)\sigma_j^B(S_j, t)\rho_{ij}^B$.
- $P(t)$ is a Poisson process with the arrival rate λ.
- $J_i(t), \quad i = 1, \ldots, d$ follows a multivariate normal distribution with mean values μ_i^J and covariance matrix $\Sigma_{ij}^J = \sigma_i^J(S_i, t)\sigma_j^J(S_j, t)\rho_{ij}^J$.
- The expected jump of the ith component is

$$\kappa_i = \mathbb{E}\left[\mathrm{e}^{J_i(t)} - 1\right] = \exp\left(\mu_i^J + \frac{1}{2}\sum_{j=1}^{d}\sigma_i^J\sigma_j^J\rho_{ij}^J\right) - 1.$$

- When $\lambda = 0$ and σ_i constant: multi Black-Scholes model.

Basket options - Problems

- For all the problems: Price u.
- For some problems also: $\Delta = \frac{\partial u}{\partial S_i}$ and $\mathcal{V} = \frac{\partial u}{\partial \sigma_i}$

1. **European spread option**

$$g(S) = \max\{S_1 - S_2 - K, 0\},$$

with settings: GBM, $S_i = 100$, $r = 0.03$, $T = 1$, $\rho = 0.5$ and $K = 5$. Two problems: constant volatility ($\sigma_i = 0.15$) or given by the function

$$\sigma_i(S_i, t) = 0.15 + 0.15(0.5 + 2t)\frac{(S_i/100 - 1.2)^2}{(S_i/100)^2 + 1.44}.$$

Basket options - Problems

2. **American put on the minimum of two assets**

$$g(S) = \max\left\{K - \min\left\{S_1, S_2\right\}, 0\right\},$$

with settings: $S_i = 40$, $r = 0.05$, $\sigma_i = 0.3$ $T = 0.5$, $\rho = 0.5$ and $K = 40$. Two problems: without jumps (Black-Scholes) or with jumps ($\mu_i^J = -0.5$, $\sigma_i^J = 0.4$, $\rho_{ij}^J = 0.5$ and $\lambda = 0.4$).

3. **Arithmetic basket options on 3 and 10 assets**

$$g(S) = \max\left\{K - \frac{1}{d}\sum_{i=1}^{d} S_i, 0\right\},$$

with settings: GBM, $S_i = 40$, $r = 0.06$, $\sigma_i = 0.2$, $T = 1$ and $K = 40$. Four problems: European/American and low constant correlation ($\rho = 0.25$), European/American high variable correlations ($\rho_{ij} = 0.9^{|i-j|}$).

Basket options - Problems

4. **European arithmetic basket options on four assets**

$$g(S) = \max\left\{K - \frac{1}{d}\sum_{i=1}^{d} S_i, 0\right\},$$

with settings: GBM, $S_i = 40$, $r = 0.06$, $\sigma_i = 0.3$, $T = 1$ and $K = 40$.
Correlation matrix:

$$\rho = \begin{pmatrix} 1 & 0.3 & 0.4 & 0.5 \\ 0.3 & 1 & 0.2 & 0.25 \\ 0.4 & 0.2 & 1 & 0.3 \\ 0.5 & 0.25 & 0.3 & 1 \end{pmatrix}.$$

Basket options - Problems

5. **European/American arithmetic basket options on five assets**

$$g(S) = \max\left\{K - \sum_{i=1}^{d} w_i S_i, 0\right\},$$

with settings: GBM, $S_i = 1$, $r = 0.05$,
$\sigma = [0.518, 0.648, 0.623, 0.570, 0.530]$,
$w = [0.381, 0.065, 0.057, 0.270, 0.227]$, $T = 1$ and $K = 1$. Correlation matrix:

$$\rho = \begin{pmatrix} 1 & 0.79 & 0.82 & 0.91 & 0.84 \\ 0.79 & 1 & 0.73 & 0.80 & 0.76 \\ 0.82 & 0.73 & 1 & 0.77 & 0.72 \\ 0.91 & 0.80 & 0.77 & 1 & 0.90 \\ 0.84 & 0.76 & 0.72 & 0.90 & 1 \end{pmatrix}.$$

- European call options.
- Three prices: in-the-money, at-the-money and out-the-money.

1. **SABR model**

   The formal definition of the SABR model reads

$$\begin{aligned} \mathrm{d}S(t) &= \sigma(t)S^{\beta}(t)\mathrm{d}W_S(t), & S(0) &= S_0 \exp(rT), \\ \mathrm{d}\sigma(t) &= \alpha\sigma(t)\mathrm{d}W_\sigma(t), & \sigma(0) &= \sigma_0, \end{aligned}$$

   where $S(t) = \bar{S}(t)\exp(r(T-t))$. Correlation between the Brownian motions, ρ. Two parameter sets:
   $T = 2,\ r = 0.0,\ S_0 = 0.5,\ \sigma_0 = 0.5,\ \alpha = 0.4,\ \beta = 0.5,\ \rho = 0$.
   $T = 10,\ r = 0.0,\ S_0 = 0.07,\ \sigma_0 = 0.4,\ \alpha = 0.8,\ \beta = 0.5,\ \rho = -0.6$.
   European call option payoff $(\max(S(T) - K_i(T), 0))$ with

$$\begin{aligned} K_i(T) &= S(0)\exp(0.1 \times \sqrt{T} \times \delta_i), \\ \delta_i &= -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5. \end{aligned}$$

Stochastic and local volatility - Problems

2. **Quadratic local stochastic volatility model**

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sqrt{V(t)}f(S(t))\mathrm{d}W_S(t),$$
$$\mathrm{d}V(t) = \kappa(\eta - V(t))\mathrm{d}t + \sigma\sqrt{V(t)}\mathrm{d}W_V(t),$$

with $f(s) = \frac{1}{2}\alpha s^2 + \beta s + \gamma$.

3. **Heston-Hull-White model**

$$\mathrm{d}S(t) = R(t)S(t)\mathrm{d}t + \sqrt{V(t)}S(t)\mathrm{d}W_S(t),$$
$$\mathrm{d}V(t) = \kappa(\eta - V(t))\mathrm{d}t + \sigma_1\sqrt{V(t)}\mathrm{d}W_V(t),$$
$$\mathrm{d}R(t) = a(b - V(t))\mathrm{d}t + \sigma_2\mathrm{d}W_R(t).$$

Our contribution

- We propose Monte Carlo-based methods.
- For Basket options: Stochastic Grid Bundling method (SGBM).
- For SABR model:
  - The mSABR simulation scheme [LGO17].
  - Multi Level Monte Carlo, MLMC, to exploit parallel features.

Stochastic Grid Bundling Method

- Early-exercise pricing method [JO15].
- Dynamic programming approach.
- Simulation and regression-based method.
- Forward in time: Monte Carlo simulation.
- Backward in time: Early-exercise policy computation.
- Step I: Generation of stochastic grid points

$$\{\mathbf{S}_{t_0}(n), \ldots, \mathbf{S}_{t_M}(n)\}, \ n = 1, \ldots, N.$$

- Step II: Option value at terminal time $t_M = T$

$$V_{t_M}(\mathbf{S}_{t_M}) = \max(h(\mathbf{S}_{t_M}), 0).$$

Stochastic Grid Bundling Method

- Backward in time, t_m, $m \leq M$,:
- Step III: Bundling into ν non-overlapping sets or partitions

$$\mathcal{B}_{t_{m-1}}(1), \ldots, \mathcal{B}_{t_{m-1}}(\nu)$$

- Step IV: Parameterizing the option values

$$Z(\mathbf{S}_{t_m}, \alpha^{\beta}_{t_m}) \approx V_{t_m}(\mathbf{S}_{t_m}).$$

- Step V: Computing the continuation and option values at t_{m-1}

$$\widehat{Q}_{t_{m-1}}(\mathbf{S}_{t_{m-1}}(n)) = \mathbb{E}[Z(\mathbf{S}_{t_m}, \alpha^{\beta}_{t_m}) | \mathbf{S}_{t_{m-1}}(n)].$$

  The option value is then given by:

$$\widehat{V}_{t_{m-1}}(\mathbf{S}_{t_{m-1}}(n)) = \max(h(\mathbf{S}_{t_{m-1}}(n)), \widehat{Q}_{t_{m-1}}(\mathbf{S}_{t_{m-1}}(n))).$$

Stochastic Grid Bundling Method

- Basis functions $\phi_1, \phi_2, \ldots, \phi_K$.
- In our case, $Z\left(\mathbf{S}_{t_m}, \alpha^{\beta}_{t_m}\right)$ depends on $\mathbf{S}_{t_m}$ only through $\phi_k(\mathbf{S}_{t_m})$:

$$Z\left(\mathbf{S}_{t_m}, \alpha^{\beta}_{t_m}\right) = \sum_{k=1}^{K} \alpha^{\beta}_{t_m}(k)\phi_k(\mathbf{S}_{t_m}).$$

- Computation of $\alpha^{\beta}_{t_m}$ (or $\widehat{\alpha}^{\beta}_{t_m}$) by least squares regression.
- The $\alpha^{\beta}_{t_m}$ determines the early-exercise policy.
- The continuation value:

$$\begin{aligned}\widehat{Q}_{t_{m-1}}(\mathbf{S}_{t_{m-1}}(n)) &= D_{t_{m-1}}\mathbb{E}\left[\left(\sum_{k=1}^{K} \widehat{\alpha}^{\beta}_{t_m}(k)\phi_k(\mathbf{S}_{t_m})\right)|\mathbf{S}_{t_{m-1}}\right] \\ &= D_{t_{m-1}}\sum_{k=1}^{K} \widehat{\alpha}^{\beta}_{t_m}(k)\mathbb{E}\left[\phi_k(\mathbf{S}_{t_m})|\mathbf{S}_{t_{m-1}}\right].\end{aligned}$$

Stochastic Grid Bundling Method

- Choosing ϕ_k: the expectations $\mathbb{E}\left[\phi_k(\mathbf{S}_{t_m})|\mathbf{S}_{t_{m-1}}\right]$ should be easy to calculate.
- The intrinsic value of the option, $h(\cdot)$, is usually an important and useful basis function. For example:
  - Geometric basket Bermudan:

$$h(\mathbf{S}_t) = \left(\prod_{\delta=1}^{d} S_t^{\delta}\right)^{\frac{1}{d}}$$

  - Arithmetic basket Bermudan:

$$h(\mathbf{S}_t) = \frac{1}{d}\sum_{\delta=1}^{d} S_{t_m}^{\delta}$$

- For $\mathbf{S}_t$ following a GBM: expectations analytically available.

Stochastic Grid Bundling Method

- SGBM has been developed as *duality-based method*.
- Provide two estimators (confidence interval).
- *Direct estimator* (high-biased estimation):

$$\widehat{V}_{t_{m-1}}(\mathbf{S}_{t_{m-1}}(n)) = \max\left(h\left(\mathbf{S}_{t_{m-1}}(n)\right), \widehat{Q}_{t_{m-1}}\left(\mathbf{S}_{t_{m-1}}(n)\right)\right),$$

$$\mathbb{E}[\widehat{V}_{t_0}(\mathbf{S}_{t_0})] = \frac{1}{N}\sum_{n=1}^{N}\widehat{V}_{t_0}(\mathbf{S}_{t_0}(n)).$$

- *Path estimator* (low-biased estimation):

$$\widehat{\tau}^*\left(\mathbf{S}(n)\right) = \min\{t_m \,:\, h\left(\mathbf{S}_{t_m}(n)\right) \geq \widehat{Q}_{t_m}\left(\mathbf{S}_{t_m}(n)\right),\ m = 1, \ldots, M\},$$

$$v(n) = h\left(\mathbf{S}_{\widehat{\tau}^*(\mathbf{S}(n))}\right),$$

$$\underline{V}_{t_0}(\mathbf{S}_{t_0}) = \lim_{N_{\mathrm{L}}}\frac{1}{N_{\mathrm{L}}}\sum_{n=1}^{N_{\mathrm{L}}} v(n).$$

Simulation of SABR model

- Simulation of the volatility process, $\sigma(t)|\sigma(s)$:

$$\sigma(t) \sim \sigma(s) \exp\left(\alpha \hat{W}_\sigma(t) - \frac{1}{2}\alpha^2(t-s)\right),$$

  where $\hat{W}_\sigma(t)$ is a independent Brownian motion.
- Simulation of the integrated variance process, $\int_s^t \sigma^2(z)\mathrm{d}z|\sigma(t), \sigma(s)$.
- Simulation of the forward process, $S(t)|S(s), \int_s^t \sigma^2(z)\mathrm{d}z, \sigma(t), \sigma(s)$.
- The conditional integrated variance is a challenging part. We propose:
  - Approximate the conditional distribution by using Fourier techniques and copulas.
  - Marginal distribution based on COS method.
  - Conditional distribution based on copulas.
  - Improvements in performance and efficiency (SCMC).

Sampling $\int_s^t \sigma^2(z)\mathrm{d}z | \sigma(t), \sigma(s)$

- It forms the basis of the mSABR method.
- Steps:
  1. Determine $F_{\log \sigma(t) | \log \sigma(s)}$ and $F_{\log \hat{Y} | \log \sigma(s)}$.
  2. Determine the correlation between $\log Y(s,t)$ and $\log \sigma(t)$.
  3. Generate correlated uniform samples, $U_{\log \sigma(t) | \log \sigma(s)}$ and $U_{\log \hat{Y} | \log \sigma(s)}$ by means of copula.
  4. From $U_{\log \sigma(t) | \log \sigma(s)}$ and $U_{\log \hat{Y} | \log \sigma(s)}$ invert original marginal distributions.
  5. The samples of $\sigma(t)|\sigma(s)$ and $Y(s,t) = \int_s^t \sigma^2(z)\mathrm{d}z | \sigma(t), \sigma(s)$ are obtained by taking exponentials.

Simulation of $S(t)|S(s), \int_s^t \sigma^2(z)\mathrm{d}z, \sigma(t), \sigma(s)$

- In the original paper, we use numerical inversion of the asset CDF.
- For the BENCHOP project, we consider an alternative scheme to take advantage of the parallel features.
- But we desire to take advantage of mSABR.
- Discretization scheme Log-Euler+ (time step Δt):

$$\begin{aligned}\log S(t+\Delta t) = \log S(t) &- \frac{1}{2} S^{2(\beta-1)}(t) \int_t^{t+\Delta t} \sigma^2(z)\mathrm{d}z \\ &+ S^{\beta-1}(t)\frac{\rho}{\alpha}\left(\sigma(t+\Delta t) - \sigma(t)\right) \\ &+ S^{\beta-1}(t)\sqrt{1-\rho^2}\int_t^{t+\Delta t} \sigma(z) dW_S(z),\end{aligned}$$

where $\int_t^{t+\Delta t} \sigma(z) dW_S(z) \sim \mathcal{N}\left(0, \int_t^{t+\Delta t} \sigma^2(z)\mathrm{d}z\right)$.

Numerical results

- Computational time vs. prescribed accuracy.
- Relative error (RE).
- For SGBM: only sequential times.
- For mSABR and MLMC: sequential times and parallel (parfor + GPU array) times.
- Computer system: Intel Core i7-4720HQ 2.6 GHz, RAM 16 Gb.

Basket options

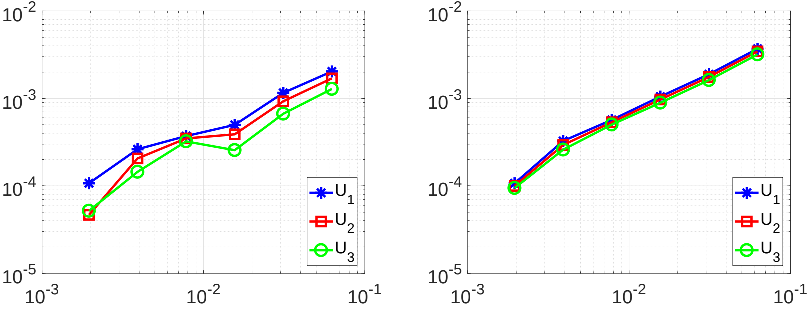
- Reference values only for Problem 5 and European options.
- Targeted precision: $< 10^{-3}$.

	Price u
3D European low corr.	28.4988
3D European high corr.	28.6025
10D European low corr.	68.7701
10D European high corr.	66.2690

Table: SGBM times(s).

Convergence of the MLMC - SABR model

- As usual for MLMC, we test the convergence of the correction estimators.



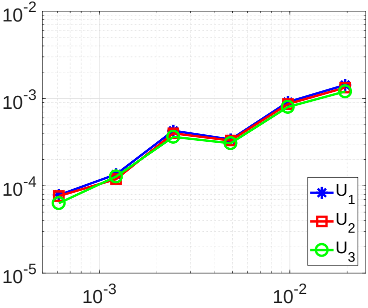
(a) Set I - Mean corrections.

(b) Set I - Var. corrections.

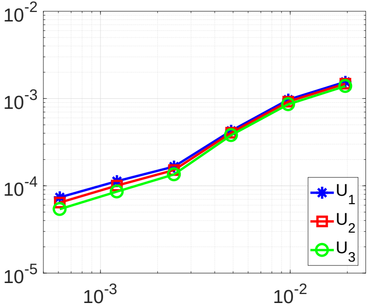
Figure: Convergence of the MLMC implementation for the SABR model.

Convergence of the MLMC - SABR model

- Similar results for Set II.



(a) Set II - Mean corrections.



(b) Set II - Var. corrections.

Figure: Convergence of the MLMC implementation for the SABR model.

SABR model

- Computational time in seconds for the considered approaches.
- Targeted precision: $< 10^{-3}$.

	Serial		Parallel	
	mSABR	MLMC	mSABR	MLMC
Set I	11.833	1.737	9.805	1.296
Set II	10.378	27.216	9.628	16.847

Table: Time (s).

Ongoing work

- Implementation of the remaining basket problems.
- Parallel version of SGBM.
- Improved parallel version of mSABR.
- MLMC + mSABR (if possible).
- Other stochastic local volatility models?

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Suggestions, comments & questions

Thank you for your attention