Asian SWIFT method

Efficient wavelet-based valuation of arithmetic Asian options

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Motivation

- Arithmetic Asian options are still attractive in financial markets, but it numerical treatment is rather challenging.
- The valuation methods relying on Fourier inversion are highly appreciated, particularly for calibration purposes, since they are extremely fast, very accurate and easy to implement.
- Lack of robustness in the existing methods (number of terms in the expansion, numerical quadratures, truncation, etc.).
- The use of wavelets for other option problems (Europeans, early-exercise, etc.) has resulted in significant improvements in this sense.
- In the context of arithmetic Asian options, SWIFT provides extra benefits.

Outline

Definitions

Option

A contract that offers the buyer the right, but not the obligation, to buy (call) or sell (put) a financial asset at an agreed-upon price (the strike price) during a certain period of time or on a specific date (exercise date). Investopedia.

Option price

The fair value to enter in the option contract. In other (mathematical) words, the (discounted) expected value of the contract.

$$v(S,t) = D(t)\mathbb{E}\left[\mathcal{P}(S(t))\right]$$

where $\mathcal{P}$ is the *payoff* function, S the underlying asset, t the exercise time and $D(t)$ the discount factor.

Definitions (II)

Pricing techniques

- Stochastic process, $S(t)$, governed by a SDE.
- Underlying models: Black-Scholes, Lévy-based, Heston, etc.
- Simulation: Monte Carlo method.
- PDEs: Feynman-Kac theorem.
- Fourier inversion techniques: characteristic function.

Types of options - payoff function

- Vanilla: involves only the value of S at exercise. Standardized.
- Exotic: involves more complicated features. Over the counter.
- Path-dependent: Asian, Barrier, Lookback, ...

Problem formulation

- In Asian derivatives, the option payoff function relies on some *average* of the underlying values at a prescribed monitoring dates.
- Thus, the final value is less volatile and the option price cheaper.
- Consider $N+1$ monitoring dates $t_i \in [0, T], i = 0, \ldots, N$.
- Where T is the maturity and $\Delta t := t_{i+1} - t_i, \forall i$ (equal-spaced).
- Assume the initial state of the price process to be known, $S(0) = S_0$.
- Let averaged price be defined as $A_N := \frac{1}{N+1}\sum_{i=0}^{N} S(t_i)$, the payoff of the *European-style* Asian call option is

$$v(S, T) = (A_N - K)^+ .$$

- The risk-neutral option valuation formula,

$$v(x, t) = \mathrm{e}^{-r(T-t)}\mathbb{E}\left[v(y, T)|x\right] = \mathrm{e}^{-r(T-t)}\int_{\mathbb{R}} v(y, T)f(y|x)\mathrm{d}y,$$

with r the risk-free rate, T the maturity, $f(y|x)$ the transitional density, typically unknown, and $v(y, T)$ the payoff function.

The SWIFT method

- A structure for wavelets in $L^2(\mathbb{R})$ is called a *multi-resolution analysis*.
- We start with a family of closed nested subspaces in $L^2(\mathbb{R})$,

$$\ldots \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \ldots, \quad \bigcap_{m\in\mathbb{Z}} \mathcal{V}_m = \{0\}, \quad \overline{\bigcup_{m\in\mathbb{Z}} \mathcal{V}_m} = L^2(\mathbb{R}),$$

where

$$f(x) \in \mathcal{V}_m \iff f(2x) \in \mathcal{V}_{m+1}.$$

- Then, it exists a function $\varphi \in \mathcal{V}_0$ generating an orthonormal basis, denoted by $\{\varphi_{m,k}\}_{k\in\mathbb{Z}}$, for each $\mathcal{V}_m$, $\varphi_{m,k}(x) = 2^{m/2}\varphi(2^m x - k)$.
- The function φ is called the *scaling function* or *father wavelet*.
- For any $f \in L^2(\mathbb{R})$, a projection map of $L^2(\mathbb{R})$ onto $\mathcal{V}_m$, denoted by $\mathcal{P}_m : L^2(\mathbb{R}) \to \mathcal{V}_m$, is defined by means of

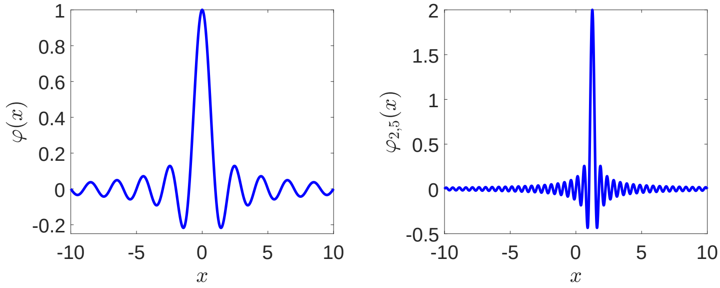
$$\mathcal{P}_m f(x) = \sum_{k\in\mathbb{Z}} c_{m,k}\varphi_{m,k}(x), \quad \text{with} \quad c_{m,k} = \langle f, \varphi_{m,k}\rangle.$$

The SWIFT method

- In this work, we employ Shannon wavelets. A set of Shannon scaling functions $\varphi_{m,k}$ in the subspace $\mathcal{V}_m$ is defined as,

$$\varphi_{m,k}(x) = 2^{m/2}\frac{\sin(\pi(2^m x - k))}{\pi(2^m x - k)} = 2^{m/2}\varphi(2^m x - k), \quad k \in \mathbb{Z},$$

  where $\varphi(z) = \mathrm{sinc}(z)$, with sinc the cardinal sine function.
- Given a function $f \in L^2(\mathbb{R})$, we will consider its expansion in terms of Shannon scaling functions at the level of resolution m.



The SWIFT method

- Our aim is to recover the coefficients $c_{m,k}$ of this approximation from the Fourier transform of the function f, denoted by $\hat{f}$, defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\xi x} f(x) \mathrm{d}x,$$

where i is the imaginary unit.
- Following wavelets theory, a function $f \in L^2(\mathbb{R})$ can be approximated at the level of resolution m by,

$$f(x) \approx \mathcal{P}_m f(x) = \sum_{k \in \mathbb{Z}} c_{m,k} \varphi_{m,k}(x),$$

where $\mathcal{P}_m f$ converges to f in $L^2(\mathbb{R})$, i.e. $\|f - \mathcal{P}_m f\|_2 \to 0, m \to +\infty$.
- The infinite series is well-approximated (see Lemma 1 of [3]) by

$$\mathcal{P}_m f(x) \approx f_m(x) := \sum_{k=k_1}^{k_2} c_{m,k} \varphi_{m,k}(x),$$

for certain accurately chosen values k_1 and k_2.

The SWIFT method

- Computation of the coefficients $c_{m,k}$: by definition,

$$c_{m,k} = \langle f, \varphi_{m,k} \rangle = \int_{\mathbb{R}} f(x)\bar{\varphi}_{m,k}(x)\mathrm{d}x = 2^{m/2} \int_{\mathbb{R}} f(x)\varphi(2^m x - k)\mathrm{d}x.$$

- By using the classical Vieta's formula,

$$\varphi(x) = \mathrm{sinc}(x) = \prod_{j=1}^{+\infty} \cos\left(\frac{\pi x}{2^j}\right).$$

- We truncate the infinite product into a finite product with J terms, then, thanks to the cosine product-to-sum identity,

$$\prod_{j=1}^{J} \cos\left(\frac{\pi x}{2^j}\right) = \frac{1}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \cos\left(\frac{2j-1}{2^J}\pi x\right).$$

- Then,

$$\int_{\mathbb{R}} f(x)\bar{\varphi}_{m,k}(x)\mathrm{d}x \approx \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \int_{\mathbb{R}} f(x)\cos\left(\frac{2j-1}{2^J}\pi\left(2^m x - k\right)\right)\mathrm{d}x.$$

The SWIFT method

- Noting that $\Re\left(\hat{f}(\xi)\right) = \int_{\mathbb{R}} f(x)\cos(\xi x)\mathrm{d}x$ and

$$\hat{f}(\xi)\mathrm{e}^{\mathrm{i}k\pi\frac{2j-1}{2^J}} = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\left(\xi x - k\pi\frac{2j-1}{2^J}\right)} f(x)\mathrm{d}x.$$

- Thus, we have,

$$c_{m,k} \approx \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \Re\left[\hat{f}\left(\frac{(2j-1)\pi 2^m}{2^J}\right) \mathrm{e}^{\frac{\mathrm{i}k\pi(2j-1)}{2^J}}\right].$$

- Putting everything together gives the following approximation of f,

$$f(x) \approx \sum_{k=k_1}^{k_2} c_{m,k}\varphi_{m,k}(x).$$

SWIFT option valuation formulas

- Truncating the integration range on $[a, b]$ in the risk-neutral valuation formula, and replacing density f by the SWIFT approximation,

$$v(x, t_0) \approx \mathrm{e}^{-rT} \sum_{k=k_1}^{k_2} c_{m,k} V_{m,k},$$

where,

$$V_{m,k} := \int_a^b v(y, T) \varphi_{m,k}(y|x) \mathrm{d}y.$$

- By employing the Vieta's formula again and interchanging summation and integration operations, we obtain that

$$V_{m,k} \approx \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \int_a^b v(y, T) \cos\left(\frac{2j-1}{2^J} \pi \left(2^m y - k\right) \right) \mathrm{d}y.$$

SWIFT option sensitivities

- Under the SWIFT framework, the estimation of the option price sensitivities, the so-called *Greeks*.
- The Greeks are defined as the partial derivatives of the option price with respect to some market/model parameter.
- They can be efficiently calculated by constructing similar series expansions.
- Generally, two possible situations can appear: the option price depends only on the parameter of interest either through the density function or payoff function.
- The partial derivative of the characteristic function and, hence, the density coefficients and the payoff function can be analytically computed for many financial models and option contracts.

SWIFT option sensitivities

- We firstly assume that the option price depends on the parameter of interest only through the density function,

$$c_{m,k}(\xi,\varsigma)=\frac{2^{m/2}}{2^{J-1}}\sum_{j=1}^{2^{J-1}}\Re\left[\hat{f}\left(\xi;\varsigma\right)\mathrm{e}^{\frac{\mathrm{i}k\xi}{2^m}}\right],$$

where $\xi=\frac{(2j-1)\pi 2^m}{2^J}$ and ς the parameter of interest.
- By differentiating (n times) the characteristic function, the "Greek" density coefficients

$$c_{m,k}^{(n)}(\xi):=\frac{\partial^n c_{m,k}(\xi,\varsigma)}{\partial\varsigma^n}=\frac{2^{m/2}}{2^{J-1}}\sum_{j=1}^{2^{J-1}}\Re\left[\frac{\partial^n\hat{f}\left(\xi;\varsigma\right)}{\partial\varsigma^n}\mathrm{e}^{\frac{\mathrm{i}k\xi}{2^m}}\right].$$

- For example, the so-called *Delta*, Δ, and *Gamma*, Γ, the first and second derivatives w.r.t. S_0, are computed by plugging the $c_{m,k}^{(n)}$,

$$\Delta:=\mathrm{e}^{-rT}\sum^{k_2}c_{m,k}^{(1)}V_{m,k},\quad\Gamma:=\mathrm{e}^{-rT}\sum^{k_2}c_{m,k}^{(2)}V_{m,k}.$$

SWIFT option sensitivities

- A second possible situation appears when the option value depends on the parameter of interest, ς, through the payoff coefficients, i.e., $V_{m,k}(\varsigma)$.
- Thus, the "Greek" payoff coefficients need to be determined by differentiating $V_{m,k}$ with respect to ς.
- Particularly, the solution for the Greeks Δ and Γ would be

$$\Delta := \mathrm{e}^{-rT} \sum_{k=k_1}^{k_2} c_{m,k} V_{m,k}^{(1)}(\varsigma), \quad \Gamma := \mathrm{e}^{-rT} \sum_{k=k_1}^{k_2} c_{m,k} V_{m,k}^{(2)}(\varsigma),$$

  where now the $c_{m,k}$ are kept invariant and $V_{m,k}^{(n)}$ represents the n-th derivative of $V_{m,k}$.
- In the context of Fourier inversion techniques, closed-form solutions for these coefficients can be usually derived.
- The case of the arithmetic Asian payoff will be addressed in the next section.

Optimal scale m, series bounds k_1 and k_2, and parameter J

- The quality in the approximation provided by the SWIFT method is affected by the scale m, the number of terms in the Vieta's approximation and the series truncation limits, k_1 and k_2.
- By Lemma 3 of [2], the error in the projection approximation of function f is bounded by

$$|f(x) - \mathcal{P}_m f(x)| \leq \frac{1}{2\pi} \int_{|\xi|>2^m\pi} \left|\hat{f}(\xi)\right| \mathrm{d}\xi.$$

- As the characteristic function, $\hat{f}$, is assumed to be known, we can compute m given a prescribed tolerance ϵ_m.
- Applying a simple quadrature rule, the error bound reads

$$\frac{1}{2\pi}\left(\left|\hat{f}(-2^m\pi)\right| + \left|\hat{f}(2^m\pi)\right|\right).$$

- More involved numerical quadratures have been tested, but the observed differences are negligible.

Optimal scale m, series bounds k_1 and k_2, and parameter J

- k_1 and k_2 can be computed based on the integration range $[a, b]$ as

$$k_1 := \lfloor 2^m a \rfloor \quad \text{and} \quad k_2 := \lceil 2^m b \rceil,$$

where m is the scale of approximation.

- Therefore we first need to choose the interval limits, a and b, in such a way that the loss of density mass is minimized.
- Cumulants-based approach,

$$[a, b] := \left[\kappa_1(Y) - L\sqrt{\kappa_2(Y) + \sqrt{\kappa_4(Y)}}, \kappa_1(Y) + L\sqrt{\kappa_2(Y) + \sqrt{\kappa_4(Y)}} \right],$$

with $\kappa_n(Y)$ representing the n-th cumulant (defined from the *cumulant-generating function*, $\mathcal{K}(\tau)$, as $\kappa_n = \mathcal{K}^{(n)}(0)$) of the random variable Y and L a constant conveniently chosen.

Optimal scale m, series bounds k_1 and k_2, and parameter J

- The dependence on m turns out to be very convenient also in the selection of the interval $[a, b]$.
- This constitutes one of the great advantages of the SWIFT method with respect to other Fourier inversion-based techniques, where a and b are arbitrarily selected.
- Thus, as we know that our approximation at scale m satisfies the tolerance ϵ_m, the error order due to the truncation should not exceed the order of ϵ_m.
- We can therefore develop an adaptive interval selection algorithm that updates the truncated range $[a, b]$ in each iteration, computes the truncation error, ϵ_τ, in the approximated density using that interval and stops when the same tolerance condition ϵ_m is prescribed.

Optimal scale m, series bounds k_1 and k_2, and parameter J

- The parameter J is then chosen to be constant (it could be selected as a function of k) based on the previously determined quantities.
- Doing so, we can benefit from the use of FFT algorithm.
- By Theorem 1 of [3], let $c^*_{m,k}$ the approximated coefficients,

$$|c_{m,k} - c^*_{m,k}| \leq 2^{m/2} \left(2\epsilon + \sqrt{2\mathcal{A}} \|f\|_2 \frac{(\pi M_{m,k})^2}{2^{2(J+1)} - (\pi M_{m,k})^2} \right),$$

assuming $J \geq log_2(\pi M_{m,k})$, and with
$M_{m,k} := \max\left(\left|2^m\mathcal{A} - k\right|, \left|2^m\mathcal{A} + k\right|\right)$, $\mathcal{A} := \max\left(|a|, |b|\right)$,
$H(x) = F(-x) + 1 - F(x)$ and $H(\mathcal{A}) < \epsilon$.

- Thus, the number of Vieta factor is selected as

$$J := \lceil \log\left(\pi M_m\right) \rceil \quad \text{with} \quad M_m := \max_{k_1 < k < k_2} M_{m,k},$$

SWIFT for Asian options under exponential Lévy models

- Exponential Lévy models: $\log S(t)$, follows a Lévy process.
- The Lévy dynamics have a stationary and i.i.d. increments and it can be written in the form

$$X(t) = \mu t + W(t) + J(t) + \lim_{\varepsilon \downarrow 0} D^{\varepsilon}(t),$$

  where W is a d-dimensional Brownian motion with covariance matrix Σ, drift vector $\mu \in \mathbb{R}^d$, J is a compound Poisson process and D^{ε} is a compensated compound Poisson process. A measure ν on $\mathbb{R}^d$ is adopted, called *Lévy measure*.
- The Lévy processes are fully determined by the *characteristic triplet* $[\Sigma, \mu, \nu]$. From the *Lévy-Khintchine* formula, the characteristic function, defined as $\hat{f}(\xi) = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi X(t)}\right]$, reads

$$\hat{f}(\xi) = \mathrm{e}^{t\vartheta(\xi)}, \quad \vartheta(\xi) = \mathrm{i}\mu \cdot \xi + \frac{1}{2}\Sigma\xi \cdot \xi + \int_{\mathbb{R}^d} \left(\mathrm{e}^{\mathrm{i}\xi \cdot x} - 1 - \mathrm{i}\xi \cdot x \mathbb{1}_{|x| \leq 1}\right) \nu(\mathrm{d}x),$$

  where ϑ is often called the *characteristic exponent*.

SWIFT for Asian options under exponential Lévy models

- The explicit representation of the characteristic function in the Lévy processes framework supposes a great advantage.
- Allows to recover the density, f, by Fourier inversion numerical techniques and price European options highly efficiently.
- The characteristic function of exponential Lévy dynamics is often available in a tractable form (ex. Black-Scholes, Merton, Variance Gamma (VG), Normal Inverse Gaussian (NIG)).
- But, for arithmetic Asian options, the derivation of the corresponding characteristic function is rather involved.
- Lets start by defining the *return* or *increment* process R_i,

$$R_i := \log\left(\frac{S(t_i)}{S(t_{i-1})}\right) \quad i = 1, \ldots, N.$$

- Based on R_i, we define a new process

$$Y_i := R_{N+1-i} + Z_{i-1}, \quad i = 2, \ldots, N,$$

where $Y_1 = R_N$ and $Z_i := \log\left(1 + \mathrm{e}^{Y_i}\right), \forall i$.

SWIFT for Asian options under exponential Lévy models

- Applying the Carverhill-Clewlow-Hodges factorization to Y_i,

$$\frac{1}{N+1}\sum_{i=0}^{N} S(t_i) = \frac{\left(1+\mathrm{e}^{Y_N}\right) S_0}{N+1}.$$

- Thus, the option price for arithmetic Asian contracts can be now expressed in terms of the transitional density of the Y_N as

$$v(x,t_0) = \mathrm{e}^{-rT}\int_{\mathbb{R}} v(y,T) f_{Y_N}(y|x)\mathrm{d}y,$$

where $x = \log S_0$ and the call payoff function is given by

$$v(y,T) = \left(\frac{S_0\left(1+\mathrm{e}^{y}\right)}{N+1} - K\right)^{+}$$

- Again, the probability density function f_{Y_N} is generally not known, even for Lévy processes. However, as the process Y_N is defined in a recursive manner, the characteristic function of Y_N can be computed iteratively as well.

Characteristic function of Y_N

- By the definition of Y_i, the initial and recursive characteristic functions are

$$\hat{f}_{Y_1}(\xi) = \hat{f}_{R_N}(\xi) = \hat{f}_R(\xi),$$
$$\hat{f}_{Y_i}(\xi) = \hat{f}_{R_{N+1-i}+Z_{i-1}}(\xi) = \hat{f}_{R_{N+1-i}}(\xi) \cdot \hat{f}_{Z_{i-1}}(\xi) = \hat{f}_R(\xi) \cdot \hat{f}_{Z_{i-1}}(\xi).$$

- By definition, the characteristic function of Z_{i-1} reads

$$\hat{f}_{Z_{i-1}}(\xi) := \mathbb{E}\left[\mathrm{e}^{-\mathrm{i}\xi \log\left(1+\mathrm{e}^{Y_{i-1}}\right)}\right] = \int_{\mathbb{R}} (1+\mathrm{e}^x)^{-\mathrm{i}\xi} f_{Y_{i-1}}(x)\mathrm{d}x.$$

- We can again apply the wavelet approximation to $f_{Y_{i-1}}$ as

$$\hat{f}_{Z_{i-1}}(\xi) \approx \int_{\mathbb{R}} (1+\mathrm{e}^x)^{-\mathrm{i}\xi} \sum_{k=k_1}^{k_2} c_{m,k}\varphi_{m,k}(x)\mathrm{d}x$$
$$= 2^{\frac{m}{2}} \sum_{k=k_1}^{k_2} c_{m,k} \int_{\mathbb{R}} (\mathrm{e}^x+1)^{-\mathrm{i}\xi} \operatorname{sinc}(2^m x - k)\, \mathrm{d}x.$$

Characteristic function of Y_N

- The integral on the right hand side needs to be computed efficiently to make the method easily implementable, robust and very fast.
- State-of-the-art methods from the literature rely on solving the integral by means of quadratures.

Theorem (Theorem 1.3.2 of [4])

Let f be defined on $\mathbb{R}$ and let its Fourier transform $\hat{f}$ be such that for some positive constant d, $|\hat{f}(\omega)| = \mathcal{O}\left(\mathrm{e}^{-d|\omega|}\right)$ for $\omega \to \pm\infty$, then as $h \to 0$

$$\frac{1}{h}\int_{\mathbb{R}} f(x)S_{j,h}(x)\mathrm{d}x - f(jh) = \mathcal{O}\left(\mathrm{e}^{-\frac{\pi d}{h}}\right),$$

where $S_{j,h}(x) = \mathrm{sinc}\left(\frac{x}{h} - j\right)$ for $j \in \mathbb{Z}$.

- The theorem above allows us to approximate the integral above provided that $g(x) := (\mathrm{e}^x + 1)^{-\mathrm{i}\xi}$ satisfies the hypothesis.

Characteristic function of Y_N

- If we consider $h = \frac{1}{2^m}$, then it follows from Theorem 1 that

$$\int_{\mathbb{R}} g(x)\mathrm{sinc}\left(2^m x - k\right)\mathrm{d}x \approx hg\left(kh\right) = \frac{1}{2^m}\left(\mathrm{e}^{\frac{k}{2^m}} + 1\right)^{-\mathrm{i}\xi}.$$

- Thus, $\hat{f}_{Z_{i-1}}$ can be approximated by

$$\hat{f}_{Z_{i-1}}(\xi) \approx 2^{-\frac{m}{2}} \sum_{k=k_1}^{k_2} c_{m,k}\left(\mathrm{e}^{\frac{k}{2^m}} + 1\right)^{-\mathrm{i}\xi}.$$

- Finally,

$$\hat{f}_{Y_i}(\xi) = \hat{f}_R(\xi)\hat{f}_{Z_{i-1}} \approx \hat{f}_R(\xi)2^{-\frac{m}{2}} \sum_{k=k_1}^{k_2} c_{m,k}\left(\mathrm{e}^{\frac{k}{2^m}} + 1\right)^{-\mathrm{i}\xi},$$

where the density coefficients $c_{m,k}$ are computed as follows

$$c_{m,k} \approx \frac{2^{m/2}}{2^{J-1}} \sum_{j=0}^{2^{J-1}} \Re\left\{\hat{f}_{Y_{i-1}}\left(\frac{(2j-1)\pi 2^m}{2^J}\right)\mathrm{e}^{\frac{\mathrm{i}k\pi(2j-1)}{2^J}}\right\}.$$

Characteristic function of Y_N

- It remains to prove that function $g(x) = (\mathrm{e}^x + 1)^{-\mathrm{i}\xi}$ satisfies $|\hat{g}(\omega)| = \mathcal{O}\left(\mathrm{e}^{-d|\omega|}\right)$ for $\omega \to \pm\infty$.
- We have derived an expression for $\hat{g}(\omega)$,

Proposition

Let $g(x) = (\mathrm{e}^x + 1)^z$, where $z = -\mathrm{i}\xi$ and $x, \xi \in \mathbb{R}$. Then,

$$\hat{g}(\omega) = \sum_{n=0}^{\infty} \binom{z}{n} \frac{2n - z}{(n - \mathrm{i}\omega)(n + \mathrm{i}(\omega + \xi))}, \quad \omega \in \mathbb{R}.$$

Proof expression $\hat{g}(\omega)$

Proposition

Let $z \in \mathbb{C}$ *and* $\binom{z}{n} = \frac{z(z-1)(z-2)\cdots(z-n+1)}{n!}$. *Then the series* $\sum_{n=0}^{\infty} \binom{z}{n} x^n$ *converges to* $(1+x)^z$ *for all complex* x *with* $|x| < 1$.

Corollary

Let $z \in \mathbb{C}$. *Then the series* $\sum_{n=0}^{\infty} \binom{z}{n} x^n y^{z-n}$ *converges to* $(x+y)^z$ *for all complex* x, y *with* $|x| < |y|$.

Proof.

The proof follows from Proposition by taking into account that $(x+y)^z = \left(y\left[\frac{x}{y}+1\right]\right)^z$. □

Proof expression $\hat{g}(\omega)$

Proof.

From the definition, we split the integral in two parts

$$\hat{g}(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega x} g(x)\mathrm{d}x = \int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}\omega x} g(x)\mathrm{d}x + \int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}\omega x} g(x)\mathrm{d}x,$$

and observe that, by Corollary above,

$$(\mathrm{e}^x + 1)^z = \sum_{n=0}^{\infty} \binom{z}{n} \mathrm{e}^{nx}, \quad \text{for } x < 0, \quad \text{and} \quad (\mathrm{e}^x + 1)^z = \sum_{n=0}^{\infty} \binom{z}{n} \mathrm{e}^{(z-n)x}, \quad \text{for } x > 0.$$

Replacing expressions and interchanging the integral and the sum, then we obtain,

$$\hat{g}(\omega) = \sum_{n=0}^{\infty} \binom{z}{n} \int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}\omega x} \mathrm{e}^{nx}\mathrm{d}x + \sum_{n=0}^{\infty} \binom{z}{n} \int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}\omega x} \mathrm{e}^{(z-n)x}\mathrm{d}x.$$

Finally, solving the integrals,

$$\hat{g}(\omega) = \sum_{n=0}^{\infty} \binom{z}{n} \frac{1}{n - \mathrm{i}\omega} + \sum_{n=0}^{\infty} \binom{z}{n} \frac{1}{n + \mathrm{i}(\omega + \xi)} = \sum_{n=0}^{\infty} \binom{z}{n} \frac{2n - z}{(n - \mathrm{i}\omega)(n + \mathrm{i}(\omega + \xi))}.$$

Proof expression $\hat{g}(\omega)$

- It is rather complicated to get a closed-form solution for the modulus of $\hat{g}(\omega)$ from this expression.
- By using Wolfram Mathematica 11.2, the infinite sum is written as

$$\hat{g}(\omega) = \frac{\xi}{2\omega+\xi}\left[\mathrm{e}^{-\pi\omega}\left(B_{-1}\left(-\mathrm{i}\omega, 1+z\right) + 2B_{-1}\left(1-\mathrm{i}\omega, z\right)\right) + \right.$$
$$+ \Gamma\left(\mathrm{i}\omega - z\right)\left(2(\mathrm{i}\omega - z)\,{}_2\tilde{F}_1\left(1-z, 1+\mathrm{i}\omega - z; 2+\mathrm{i}\omega - z; -1\right) - \right.$$
$$\left.\left. - \ {}_2\tilde{F}_1\left(-z, \mathrm{i}\omega - z; 1+\mathrm{i}\omega - z; -1\right)\right)\right],$$

  in terms of gamma, Γ, beta, B, and regularized hypergeometric, ${}_2\tilde{F}_1(a, b; c; \nu)$. (Modulus represented in next slide).
- The shape of $|\hat{g}(\omega)|$ does not depend on the value given to ξ.
- Different ξ just originates a shift of the same function.
- The two peaks observed in the plot correspond to the poles of $\hat{g}(\omega)$ located at $\omega = 0$ and $\omega = -\xi$.
- $\hat{g}(\omega)$ presents a symmetry at $\omega = -\xi/2$ (it is straightforward to see that $\hat{g}(\omega - \xi/2) = \hat{g}(-\omega - \xi/2)$).

“Proof” modulus of $\hat{g}(\omega)$

- Representing $|\hat{g}(\omega)|$,

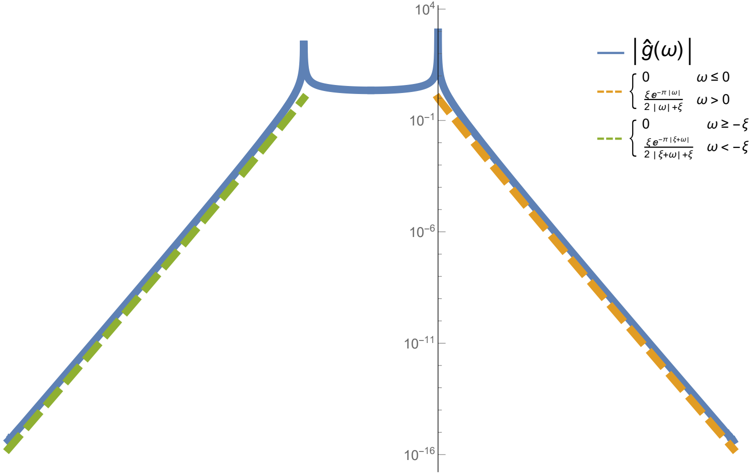


Figure: Modulus of $\hat{g}(\omega)$.

General application

- The following theorem generalises the results stated in the previous Theorem. Thus, it can be applied under weaker conditions on the decay of $|\hat{g}(\omega)|$.

Theorem

Let f be defined on $\mathbb{R}$ and let $\hat{f}$ be its Fourier transform. Then,

$$\left|\frac{1}{h}\int_{\mathbb{R}} f(x)S_{j,h}(x)\mathrm{d}x - f(jh)\right| \leq \frac{1}{2\pi}\int_{|\omega|>\frac{\pi}{h}} \left|\hat{f}(\omega)\right| \mathrm{d}\omega,$$

where $S_{j,h}(x) = \mathrm{sinc}\left(\frac{x}{h} - j\right)$ for $j \in \mathbb{Z}$.

General application - proof

Proof.

As mentioned in Lemma 3 of [2], the approximation error $|f(x) - \mathcal{P}_m f(x)|$ is uniformly bounded for all $x \in \mathbb{R}$,

$$|f(x) - \mathcal{P}_m f(x)| \leq \frac{1}{2\pi} \int_{|\omega|>2^m\pi} \left|\hat{f}(\omega)\right| \mathrm{d}\omega,$$

where $\mathcal{P}_m f(x) = \sum_{k\in\mathbb{Z}} c_{m,k}\varphi_{m,k}(x)$. In particular, this is valid for $x = jh$ with $h = 1/2^m$,

$$|f(jh) - \mathcal{P}_m f(jh)| \leq \frac{1}{2\pi} \int_{|\omega|>2^m\pi} \left|\hat{f}(\omega)\right| \mathrm{d}\omega.$$

We observe that

$$\mathcal{P}_m f(jh) = \sum_{k\in\mathbb{Z}} c_{m,k}\varphi_{m,k}(jh) = \sum_{k\in\mathbb{Z}} c_{m,k} 2^{m/2}\varphi(j-k),$$

where $\varphi(j-k) = \delta_{jk}$, and δ_{jk} is the Kronecker delta and then $\mathcal{P}_m f(jh) = 2^{m/2} c_{m,j}$. Finally, if we take into account that $c_{m,j} = \int_{\mathbb{R}} f(x)\varphi_{m,j}(x)\mathrm{d}x$. Thus,

$$\mathcal{P}_m f(jh) = 2^{m/2} \cdot 2^{m/2} \int_{\mathbb{R}} f(x)\varphi(2^m x - j)\mathrm{d}x = 2^m \int_{\mathbb{R}} f(x)\mathrm{sinc}(2^m x - j)\mathrm{d}x,$$

and this concludes the proof since $2^m = 1/h$.

Error bound of $\hat{f}_{Z_{i-1}}(\xi)$

- The error committed in the approximation of $\hat{f}_{Z_{i-1}}(\xi)$ is bounded.

Proposition

Let $\mathcal{F}_{Z_{i-1}}(\xi)$, $\mathcal{G}_{Z_{i-1}}(\xi)$ and $\mathcal{E}(\xi)$ be defined as follows,

$$\mathcal{F}_{Z_{i-1}}(\xi) = 2^{\frac{m}{2}} \sum_{k=k_1}^{k_2} c_{m,k} \int_{\mathbb{R}} (\mathrm{e}^x + 1)^{-\mathrm{i}\xi} \operatorname{sinc}(2^m x - k) \, \mathrm{d}x,$$

$$\mathcal{G}_{Z_{i-1}}(\xi) = 2^{-\frac{m}{2}} \sum_{k=k_1}^{k_2} c_{m,k} \left(\mathrm{e}^{\frac{k}{2^m}} + 1\right)^{-\mathrm{i}\xi},$$

and the difference, $\mathcal{E}(\xi) = \mathcal{F}_{Z_{i-1}}(\xi) - \mathcal{G}_{Z_{i-1}}(\xi)$.
Then, $|\mathcal{E}(\xi)|$ is uniformly bounded by

$$|\mathcal{E}(\xi)| \leq \mathcal{C}(k_2 - k_1 + 1)\mathrm{e}^{-\pi^2 2^m}$$

where $\mathcal{C}$ is a constant.

Error bound of $\hat{f}_{Z_{i-1}}(\xi)$ - proof

Proof.

We observe that,

$$\mathcal{E}(\xi) = 2^{-\frac{m}{2}} \sum_{k=k_1}^{k_2} c_{m,k} \left[2^m \int_{\mathbb{R}} (\mathrm{e}^x + 1)^{-\mathrm{i}\xi} \,\mathrm{sinc}\,(2^m x - k)\,\mathrm{d}x - \left(\mathrm{e}^{\frac{k}{2^m}} + 1 \right)^{-\mathrm{i}\xi} \right].$$

Then, by Theorem 1 with $d = \pi$,

$$|\mathcal{E}(\xi)| \leq 2^{-\frac{m}{2}} \mathcal{C} \sum_{k=k_1}^{k_2} |c_{m,k}| \mathrm{e}^{-\pi^2 2^m},$$

for a certain constant $\mathcal{C}$. The proposition holds by taking into account that,

$$|c_{m,k}| \leq \int_{\mathbb{R}} f(x) |\varphi_{m,k}(x)| \mathrm{d}x \leq 2^{\frac{m}{2}},$$

where the last inequality is satisfied since f is a density function and $|\varphi_{m,k}(x)| \leq 2^{\frac{m}{2}}$. □

Payoff coefficients

- To complete the SWIFT pricing formula, compute the payoff coefficients, $V_{m,k}$,

$$V_{m,k} = \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \left[\frac{S_0}{N+1} \left(I_2^{j,k}(\tilde{x}, b) + I_0^{j,k}(\tilde{x}, b) \right) - K I_0^{j,k}(\tilde{x}, b) \right],$$

where $\tilde{x} = \log\left(\frac{K(N+1)}{S_0} - 1\right)$ and the functions $I_0^{j,k}$ and $I_2^{j,k}$ are defined by the following integrals

$$I_0^{j,k}(x_1, x_2) := \int_{x_1}^{x_2} \cos\left(C_j\left(2^m y - k\right)\right) \mathrm{d}y,$$

$$I_2^{j,k}(x_1, x_2) := \int_{x_1}^{x_2} \mathrm{e}^y \cos\left(C_j\left(2^m y - k\right)\right) \mathrm{d}y,$$

with $C_j = \frac{2j-1}{2^J}\pi$. These integrals are analytically available.

"Greek" coefficients

- We consider Δ and Γ. In the context of Asian options under Lévy processes, only the payoff coefficients, $V_{m,k}$ are affected by S_0. Thus, by differentiating $V_{m,k}$ with respect to S_0, we obtain

$$V_{m,k}^{(1)} = \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \left[\frac{I_2^{j,k}(\tilde{x}, b) + I_0^{j,k}(\tilde{x}, b)}{N+1} + \frac{S_0 \left(\frac{\partial I_2^{j,k}(\tilde{x}, b)}{\partial S_0} + \frac{\partial I_0^{j,k}(\tilde{x}, b)}{\partial S_0} \right)}{N+1} - K \frac{\partial I_0^{j,k}(\tilde{x}, b)}{\partial S_0} \right].$$

- Applying the chain rule, the partial derivatives of $I_u^{j,k}$, $u \in \{0, 2\}$,

$$\frac{\partial I_u^{j,k}(\tilde{x}, b)}{\partial S_0} = \frac{\partial I_u^{j,k}(\tilde{x}, b)}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial S_0}, \quad \frac{\partial I_u^{j,k}(a, \tilde{x})}{\partial S_0} = \frac{\partial I_u^{j,k}(a, \tilde{x})}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial S_0},$$

where

$$\frac{\partial \tilde{x}}{\partial S_0} = -\frac{K(N+1)}{S_0 K(N+1) - S_0^2},$$

and $\frac{\partial I_u^{j,k}(\tilde{x},b)}{\partial \tilde{x}}$ and $\frac{\partial I_u^{j,k}(a,\tilde{x})}{\partial \tilde{x}}$ have analytic solution.

- Following the same procedure, a closed-form solution can be similarly derived for the second derivative of $V_{m,k}$.

Numerical results

- We compare the SWIFT method against a state-of-the-art method, the well-known COS method, particularly the COS variant for arithmetic Asian option, called *ASCOS* method [5].
- To the best of our knowledge, the ASCOS method provides the best balance between accuracy and efficiency.
- Arithmetic Asian call option valuation with varying number of monitoring dates, $N = 12$ (monthly), $N = 50$ (weekly) and $N = 250$ (daily), and conceptually different underlying Lévy dynamics: *Geometric Brownian motion* (GBM) and *Normal inverse Gaussian* (NIG).
- We assess not only the accuracy in the solution but also the computational performance.
- All the experiments have been conducted in a computer system with the following characteristics: CPU Intel Core i7-4720HQ 2.6GHz and memory of 16GB RAM. The employed software package is Matlab R2017b.

Reference GBM

# Decimals	Method	$N = 12$	$N = 50$	$N = 250$
4	ASCOS SWIFT	$N_c = 128,\ n_q = 200$ $m = 5$	$N_c = 128,\ n_q = 200$ $m = 6$	$N_c = 128,\ n_q = 200$ $m = 7$
6	ASCOS SWIFT	$N_c = 144,\ n_q = 225$ $m = 5$	$N_c = 384,\ n_q = 600$ $m = 6$	$N_c = 384,\ n_q = 600$ $m = 7$
8	ASCOS SWIFT	$N_c = 192,\ n_q = 300$ $m = 5$	$N_c = 384,\ n_q = 600$ $m = 6$	$N_c = 768,\ n_q = 1200$ $m = 8$
10	ASCOS SWIFT	$N_c = 256,\ n_q = 400$ $m = 6$	$N_c = 512,\ n_q = 800$ $m = 7$	$N_c = 5120,\ n_q = 8000$ $m = 8$

Table: GBM. The reference values are 11.9049157487 ($N = 12$), 11.9329382045 ($N = 50$) and 11.9405631571 ($N = 250$).

Reference NIG

# Decimals	Method	$N = 12$	$N = 50$	$N = 250$
1	ASCOS SWIFT	$N_c = 64,\ n_q = 100$ $m = 5$	$N_c = 128,\ n_q = 200$ $m = 5$	$N_c = 128,\ n_q = 200$ $m = 4$
2	ASCOS SWIFT	$N_c = 128,\ n_q = 200$ $m = 6$	$N_c = 128,\ n_q = 200$ $m = 5$	$N_c = 192,\ n_q = 300$ $m = 5$
3	ASCOS SWIFT	$N_c = 128,\ n_q = 200$ $m = 6$	$N_c = 192,\ n_q = 300$ $m = 5$	$N_c = 192,\ n_q = 300$ $m = 7$
4	ASCOS SWIFT	$N_c = 256,\ n_q = 400$ $m = 7$	$N_c = 256,\ n_q = 400$ $m = 8$	$N_c = 512,\ n_q = 800$ $m = 9$

Table: NIG. The reference values are 1.0135 ($N = 12$), 1.0377 ($N = 50$) and 1.0444 ($N = 250$).

Results on GBM

GBM		$N = 12$	$N = 50$	$N = 250$
		ASCOS		
$N_c = 64,\ n_q = 100$	Error Time (sec.)	3.75×10^{-4} 0.03	8.34×10^{-4} 0.02	7.17×10^{-3} 0.01
$N_c = 128,\ n_q = 200$	Error Time (sec.)	8.37×10^{-7} 0.03	7.43×10^{-6} 0.02	3.82×10^{-5} 0.02
$N_c = 256,\ n_q = 400$	Error Time (sec.)	= 0.16	5.33×10^{-7} 0.12	1.58×10^{-7} 0.11
$N_c = 512,\ n_q = 800$	Error Time (sec.)	= 1.96	= 1.80	3.04×10^{-8} 1.85
$N_c = 1024,\ n_q = 1600$	Error Time (sec.)	= 13.99	= 13.99	= 14.25
		SWIFT		
$m = 4$	Error Time (sec.)	2.70×10^{-4} 0.01	1.27×10^{-2} 0.01	3.82×10^{-2} 0.03
$m = 5$	Error Time (sec.)	7.47×10^{-9} 0.01	9.78×10^{-5} 0.02	4.01×10^{-3} 0.06
$m = 6$	Error Time (sec.)	= 0.02	3.55×10^{-10} 0.10	6.96×10^{-4} 0.40
$m = 7$	Error Time (sec.)	= 0.08	= 0.34	1.21×10^{-8} 1.37
$m = 8$	Error Time (sec.)	= 0.33	= 1.31	= 5.11

Table: SWIFT vs. ASCOS. Setting: **GBM**, $S_0 = 100$, $r = 0.0367$, $\sigma = 0.17801$, $T = 1$ and $K = 90$. The reference values are 11.9049157487 ($N = 12$), 11.9329382045 ($N = 50$) and 11.9405631571 ($N = 250$).

Results on NIG

NIG		$N = 12$	$N = 50$	$N = 250$
			ASCOS	
$N_c = 64,\ n_q = 100$	Abs error CPU time	7.78×10^{-3} 0.03	1.71×10^{-1} 0.03	8.75×10^{-2} 0.02
$N_c = 128,\ n_q = 200$	Abs error CPU time	2.60×10^{-4} 0.03	5.89×10^{-3} 0.03	1.49×10^{-2} 0.03
$N_c = 256,\ n_q = 400$	Abs error CPU time	= 0.19	= 0.17	1.42×10^{-4} 0.15
$N_c = 512,\ n_q = 800$	Abs error CPU time	= 1.98	= 1.96	= 2.02
$N_c = 1024,\ n_q = 1600$	Abs error CPU time	= 14.38	= 14.22	= 14.71
			SWIFT	
$m = 4$	Abs error CPU time	9.72×10^{-2} 0.02	9.27×10^{-2} 0.02	4.01×10^{-2} 0.04
$m = 5$	Abs error CPU time	5.69×10^{-3} 0.02	6.92×10^{-4} 0.03	4.50×10^{-3} 0.08
$m = 6$	Abs error CPU time	2.13×10^{-4} 0.02	9.12×10^{-4} 0.12	9.11×10^{-4} 0.48
$m = 7$	Abs error CPU time	= 0.13	= 0.47	= 1.52
$m = 8$	Abs error CPU time	= 0.39	= 1.46	= 5.85

Table: SWIFT vs. ASCOS. Setting: **NIG**, $S_0 = 100$, $r = 0.0367$, $\sigma = 0.0$, $\alpha = 6.1882$, $\beta = -3.8941$, $\delta = 0.1622$, $T = 1$ and $K = 110$. The reference values are 1.0135 ($N = 12$), 1.0377 ($N = 50$) and 1.0444 ($N = 250$).

Results Greeks

Strike	Method	$K = 80\%$	$K = 100\%$	$K = 120\%$
			GBM	
Δ	SWIFT	0.97573	0.57645	0.05984
	Ref.	0.97519	0.57036	0.05979
Γ	SWIFT	0.00182	0.03788	0.01123
	Ref.	0.00181	0.03777	0.01145
			NIG	
Δ	SWIFT	0.95132	0.67561	0.03562
	Ref.	0.95015	0.67220	0.03503
Γ	SWIFT	0.00268	0.03639	0.00716
	Ref.	0.00272	0.03617	0.00733

Table: Option sensitivities, Greeks Δ and Γ. Strike K as a % of S_0. Setting: $N = 12$, $m = 6$.

Conclusions

- A new Fourier inversion-based technique has been proposed in the framework of discretely monitored Asian options under exponential Lévy processes.
- The application of SWIFT to the Asian pricing problem allows to overcome the main drawbacks attributed to this type of methods.
- Specially, SWIFT allows to avoid the numerical integration in the recovery of the characteristic function.
- SWIFT results in a highly accurate and fast technique, outperforming the competitors in most of the analysed situations.

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Acknowledgements & Questions

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