Asian SWIFT method
Efficient wavelet-based valuation of arithmetic Asian options

Álvaro Leitao, Luis Ortiz-Gracia and Emma I. Wagner

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Motivation

- Arithmetic Asian options are still attractive in financial markets, but its numerical treatment is rather challenging.
- The valuation methods relying on Fourier inversion are highly appreciated, particularly for calibration purposes, since they are extremely fast, very accurate and easy to implement.
- Lack of robustness in the existing methods (number of terms in the expansion, numerical quadratures, truncation, etc.).
- The use of wavelets for other option problems (Europeans, early-exercise, etc.) has resulted in significant improvements in this sense.
- In the context of arithmetic Asian options, SWIFT provides extra benefits.
1. Problem formulation

2. The SWIFT method

3. SWIFT for Asian options

4. Numerical results

5. Conclusions
Problem formulation

- In Asian derivatives, the option payoff function relies on some *average* of the underlying values at a prescribed monitoring dates.
- Thus, the final value is less volatile and the option price cheaper.
- Consider $N + 1$ monitoring dates $t_i \in [0, T], i = 0, \ldots, N$.
- Where $T$ is the maturity and $\Delta t := t_{i+1} - t_i, \forall i$ (equal-spaced).
- Assume the initial state of the price process to be known, $S(0) = S_0$.
- Let averaged price be defined as $A_N := \frac{1}{N+1} \sum_{i=0}^{N} S(t_i)$, the payoff of the *European-style* Asian call option is
  $$ v(S, T) = (A_N - K)^+.$$

- The risk-neutral option valuation formula,
  $$ v(x, t) = e^{-r(T-t)} \mathbb{E} [v(y, T)|x] = e^{-r(T-t)} \int_{\mathbb{R}} v(y, T) f(y|x) \, dy,$$
  with $r$ the risk-free rate, $T$ the maturity, $f(y|x)$ the transitional density, typically unknown, and $v(y, T)$ the payoff function.
The SWIFT method

- A structure for wavelets in $L^2(\mathbb{R})$ is called a *multi-resolution analysis*.
- We start with a family of closed nested subspaces in $L^2(\mathbb{R})$,

$$\ldots \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \ldots, \quad \bigcap_{m \in \mathbb{Z}} \mathcal{V}_m = \{0\}, \quad \bigcup_{m \in \mathbb{Z}} \mathcal{V}_m = L^2(\mathbb{R}),$$

where

$$f(x) \in \mathcal{V}_m \iff f(2x) \in \mathcal{V}_{m+1}.$$  

- Then, it exists a function $\varphi \in \mathcal{V}_0$ generating an orthonormal basis, denoted by $\{\varphi_{m,k}\}_{k \in \mathbb{Z}}$, for each $\mathcal{V}_m$, $\varphi_{m,k}(x) = 2^{m/2} \varphi(2^m x - k)$.
- The function $\varphi$ is called the *scaling function* or *father wavelet*.
- For any $f \in L^2(\mathbb{R})$, a projection map of $L^2(\mathbb{R})$ onto $\mathcal{V}_m$, denoted by $\mathcal{P}_m : L^2(\mathbb{R}) \rightarrow \mathcal{V}_m$, is defined by means of

$$\mathcal{P}_mf(x) = \sum_{k \in \mathbb{Z}} c_{m,k} \varphi_{m,k}(x), \quad \text{with} \quad c_{m,k} = \langle f, \varphi_{m,k} \rangle.$$
In this work, we employ Shannon wavelets. A set of Shannon scaling functions $\varphi_{m,k}$ in the subspace $V_m$ is defined as,

$$
\varphi_{m,k}(x) = 2^{m/2} \frac{\sin(\pi(2^m x - k))}{\pi(2^m x - k)} = 2^{m/2} \varphi(2^m x - k), \quad k \in \mathbb{Z},
$$

where $\varphi(z) = \text{sinc}(z)$, with $\text{sinc}$ the cardinal sine function.

Given a function $f \in L^2(\mathbb{R})$, we will consider its expansion in terms of Shannon scaling functions at the level of resolution $m$.

Our aim is to recover the coefficients $c_{m,k}$ of this approximation from the Fourier transform of the function $f$, denoted by $\hat{f}$, defined as

$$
\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) dx,
$$

where $i$ is the imaginary unit.
The SWIFT method

- Following wavelets theory, a function $f \in L^2(\mathbb{R})$ can be approximated at the level of resolution $m$ by,

\[ f(x) \approx \mathcal{P}_m f(x) = \sum_{k \in \mathbb{Z}} c_{m,k} \varphi_{m,k}(x), \]

where $\mathcal{P}_m f$ converges to $f$ in $L^2(\mathbb{R})$, i.e. $\|f - \mathcal{P}_m f\|_2 \to 0$, when $m \to +\infty$.

- The infinite series is well-approximated (see Lemma 1 of [2]) by a finite summation,

\[ \mathcal{P}_m f(x) \approx f_m(x) := \sum_{k=k_1}^{k_2} c_{m,k} \varphi_{m,k}(x), \]

for certain accurately chosen values $k_1$ and $k_2$. 

The SWIFT method

- Computation of the coefficients $c_{m,k}$: by definition,

$$c_{m,k} = \langle f, \varphi_{m,k} \rangle = \int_{\mathbb{R}} f(x) \overline{\varphi}_{m,k}(x) \, dx = 2^{m/2} \int_{\mathbb{R}} f(x) \varphi(2^{m}x - k) \, dx.$$

- Using the classical Vieta’s formula truncated with $2^{J-1}$ terms, the cosine product-to-sum identity and the definition of the characteristic function, the coefficients, $c_{m,k}$, can be approximated by

$$c_{m,k} \approx \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \Re \left[ \hat{f} \left( \frac{(2j-1)\pi 2^{m}}{2^{J}} \right) e^{\frac{ik\pi(2j-1)}{2^{J}}} \right].$$

- Putting everything together gives the following approximation of $f$

$$f(x) \approx \sum_{k=k_1}^{k_2} c_{m,k} \varphi_{m,k}(x).$$
Truncating the integration range on \([a, b]\) and replacing density \(f\) by the SWIFT approximation,

\[
\nu(x, t_0) \approx e^{-rT} \sum_{k=k_1}^{k_2} c_{m,k} V_{m,k},
\]

where,

\[
V_{m,k} := \int_a^b \nu(y, T) \varphi_{m,k}(y|x) dy.
\]

By employing the Vieta’s formula again and interchanging summation and integration operations, we obtain that

\[
V_{m,k} \approx \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \int_a^b \nu(y, T) \cos \left( \frac{2j - 1}{2^J} \pi \left( 2^m y - k \right) \right) dy.
\]
Exponential Lévy models: $\log S(t)$, follows a Lévy process.

The Lévy dynamics have a stationary and i.i.d. increments, fully described from its characteristic function.

But, for arithmetic Asian options, the derivation of the corresponding characteristic function is rather involved.

Let's start by defining the return or increment process $R_i$,

$$R_i := \log \left( \frac{S(t_i)}{S(t_{i-1})} \right) \quad i = 1, \ldots, N.$$ 

Based on $R_i$, we define a new process

$$Y_i := R_{N+1-i} + Z_{i-1}, \quad i = 2, \ldots, N,$$

where $Y_1 = R_N$ and $Z_i := \log (1 + e^{Y_i})$, $\forall i$. 
Applying the Carverhill-Clewlow-Hodges factorization to $Y_i$,

$$\frac{1}{N+1} \sum_{i=0}^{N} S(t_i) = \frac{(1 + e^{Y_N}) S_0}{N+1}.$$ 

Thus, the option price for arithmetic Asian contracts can be now expressed in terms of the transitional density of the $Y_N$ as

$$v(x, t_0) = e^{-rT} \int_{\mathbb{R}} v(y, T) f_{Y_N}(y) \, dy,$$

where $x = \log S_0$ and the call payoff function is given by

$$v(y, T) = \left( \frac{S_0 (1 + e^y)}{N+1} - K \right)^+$$

Again, the probability density function $f_{Y_N}$ is generally not known, even for Lévy processes. However, as the process $Y_N$ is defined in a recursive manner, the characteristic function of $Y_N$ can be computed iteratively as well.
Characteristic function of $Y_N$

- By the definition of $Y_i$, the initial and recursive characteristic functions are
  $$\hat{f}_{Y_1}(\xi) = \hat{f}_{R_N}(\xi) = \hat{f}_R(\xi),$$
  $$\hat{f}_{Y_i}(\xi) = \hat{f}_{R_{N+1-i} + Z_{i-1}}(\xi) = \hat{f}_{R_{N+1-i}}(\xi) \cdot \hat{f}_{Z_{i-1}}(\xi) = \hat{f}_R(\xi) \cdot \hat{f}_{Z_{i-1}}(\xi).$$

- By definition, the characteristic function of $Z_{i-1}$ reads
  $$\hat{f}_{Z_{i-1}}(\xi) := \mathbb{E} \left[ e^{-i\xi \log(1+e^{Y_{i-1}})} \right] = \int_{\mathbb{R}} (1 + e^x)^{-i\xi} f_{Y_{i-1}}(x) dx.$$

- We can again apply the wavelet approximation to $f_{Y_{i-1}}$ as
  $$\hat{f}_{Z_{i-1}}(\xi) \approx \int_{\mathbb{R}} (1 + e^x)^{-i\xi} \sum_{k=1}^{k_2} c_{m,k} \varphi_{m,k}(x) dx$$
  $$= 2^m \sum_{k=k_1}^{k_2} c_{m,k} \int_{\mathbb{R}} (e^x + 1)^{-i\xi} \text{sinc} \left( 2^m x - k \right) dx.$$
Characteristic function of $Y_N$

- The integral on the right hand side needs to be computed efficiently to make the method easily implementable, robust and very fast.
- State-of-the-art methods from the literature rely on solving the integral by means of quadratures.

**Theorem (Theorem 1.3.2 of [3])**

Let $f$ be defined on $\mathbb{R}$ and let its Fourier transform $\hat{f}$ be such that for some positive constant $d$, $|\hat{f}(\omega)| = \mathcal{O}(e^{-d|\omega|})$ for $\omega \to \pm \infty$, then as $h \to 0$

$$\frac{1}{h} \int_{\mathbb{R}} f(x)S_{j,h}(x)\,dx - f(jh) = \mathcal{O}(e^{-\frac{\pi d}{h}}),$$

where $S_{j,h}(x) = \text{sinc} \left( \frac{x}{h} - j \right)$ for $j \in \mathbb{Z}$.

- Theorem 1 allows us to approximate the integral above provided that $g(x) := (e^x + 1)^{-i\xi}$ satisfies the hypothesis.
Characteristic function of $Y_N$

- If we consider $h = \frac{1}{2^m}$, then it follows from Theorem 1 that
  \[
  \int_{\mathbb{R}} g(x) \text{sinc} \left( 2^m x - k \right) \, dx \approx h g \left( kh \right) = \frac{1}{2^m} \left( e^{\frac{k}{2^m}} + 1 \right)^{-i\xi}.
  \]

- Thus, $\hat{f}_{Z_{i-1}}$ can be approximated by
  \[
  \hat{f}_{Z_{i-1}}(\xi) \approx 2^{-\frac{m}{2}} \sum_{k=k_1}^{k_2} c_{m,k} \left( e^{\frac{k}{2^m}} + 1 \right)^{-i\xi}.
  \]

- Finally,
  \[
  \hat{f}_{Y_i}(\xi) = \hat{f}_R(\xi) \hat{f}_{Z_{i-1}} \approx \hat{f}_R(\xi) 2^{-\frac{m}{2}} \sum_{k=k_1}^{k_2} c_{m,k} \left( e^{\frac{k}{2^m}} + 1 \right)^{-i\xi},
  \]
  where the density coefficients $c_{m,k}$ are computed as follows
  \[
  c_{m,k} \approx \frac{2^m/2}{2^{J-1}} \sum_{j=0}^{2^{J-1}} \Re \left\{ \hat{f}_{Y_{i-1}} \left( \frac{(2j - 1)\pi 2^m}{2^J} \right) e^{\frac{ik\pi(2j-1)}{2^J}} \right\}.
  \]
It remains to prove that function \( g(x) = (e^x + 1)^{-i\xi} \) satisfies
\[
|\hat{g}(\omega)| = O \left( e^{-d|\omega|} \right) \text{ for } \omega \to \pm\infty.
\]
We have derived an expression for \( \hat{g}(\omega) \),

**Proposition**

Let \( g(x) = (e^x + 1)^z \), where \( z = -i\xi \) and \( x, \xi \in \mathbb{R} \). Then,

\[
\hat{g}(\omega) = \sum_{n=0}^{\infty} \binom{z}{n} \frac{2n - z}{(n - i\omega)(n + i(\omega + \xi))}, \quad \omega \in \mathbb{R}.
\]

It is rather complicated to get a closed-form solution for the modulus of \( \hat{g}(\omega) \) from this expression.
By employing Wolfram Mathematica 11.2, the infinite sum is written as

\[
\hat{g}(\omega) = \frac{\xi}{2\omega + \xi} \left[ e^{-\pi\omega} \left( B_{-1} (-i\omega, 1 + z) + 2B_{-1} (1 - i\omega, z) \right) + \\
+ \Gamma (i\omega - z) \left( 2(i\omega - z) \tilde{F}_1 (1 - z, 1 + i\omega - z; 2 + i\omega - z; -1) - \\
- 2 \tilde{F}_1 (-z, i\omega - z; 1 + i\omega - z; -1) \right) \right],
\]

in terms of gamma, \( \Gamma \), beta, \( B \), and regularized hypergeometric, \( \tilde{F}_1(a, b; c; \nu) \).
Characteristic function of $Y_N$

- Representing $|\hat{g}(\omega)|$,

**Figure**: Modulus of $\hat{g}(\omega)$. 
Payoff coefficients

To complete the SWIFT pricing formula, compute the payoff coefficients, \( V_{m,k} \),

\[
V_{m,k} = \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \left[ \frac{S_0}{N + 1} \left( I_{2}^{j,k}(\tilde{x}, b) + I_{0}^{j,k}(\tilde{x}, b) \right) - K I_{0}^{j,k}(\tilde{x}, b) \right],
\]

where \( \tilde{x} = \log \left( \frac{K(N+1)}{S_0} - 1 \right) \) and the functions \( I_{0}^{j,k} \) and \( I_{2}^{j,k} \) are defined by the following integrals

\[
I_{0}^{j,k}(x_1, x_2) := \int_{x_1}^{x_2} \cos \left( C_j \left( 2^m y - k \right) \right) \, dy,
\]

\[
I_{2}^{j,k}(x_1, x_2) := \int_{x_1}^{x_2} e^y \cos \left( C_j \left( 2^m y - k \right) \right) \, dy,
\]

with \( C_j = \frac{2j-1}{2^J} \pi \). These integrals are analytically available.
Numerical results

- We compare the SWIFT method against a state-of-the-art method, the well-known COS method, particularly the COS variant for arithmetic Asian option, called ASCOS method [4].
- To the best of our knowledge, the ASCOS method provides the best balance between accuracy and efficiency.
- Arithmetic Asian call option valuation with varying number of monitoring dates, $N = 12$ (monthly), $N = 50$ (weekly) and $N = 250$ (daily), and conceptually different underlying Lévy dynamics: Geometric Brownian motion (GBM) and Normal inverse Gaussian (NIG).
- We assess not only the accuracy in the solution but also the computational performance.
- All the experiments have been conducted in a computer system with the following characteristics: CPU Intel Core i7-4720HQ 2.6GHz and memory of 16GB RAM. The employed software package is Matlab R2017b.
## Results on GBM

<table>
<thead>
<tr>
<th>GBM</th>
<th>N = 12</th>
<th>N = 50</th>
<th>N = 250</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ASCOS</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N_c = 64, n_q = 100$</td>
<td>Error</td>
<td>$3.75 \times 10^{-4}$</td>
<td>$8.34 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>Time (sec.)</td>
<td>0.03</td>
<td>0.02</td>
</tr>
<tr>
<td>$N_c = 128, n_q = 200$</td>
<td>Error</td>
<td>$8.37 \times 10^{-7}$</td>
<td>$7.43 \times 10^{-6}$</td>
</tr>
<tr>
<td></td>
<td>Time (sec.)</td>
<td>0.03</td>
<td>0.02</td>
</tr>
<tr>
<td>$N_c = 256, n_q = 400$</td>
<td>Error</td>
<td>=</td>
<td>$5.33 \times 10^{-7}$</td>
</tr>
<tr>
<td></td>
<td>Time (sec.)</td>
<td>0.16</td>
<td>0.12</td>
</tr>
<tr>
<td>$N_c = 512, n_q = 800$</td>
<td>Error</td>
<td>=</td>
<td>=</td>
</tr>
<tr>
<td></td>
<td>Time (sec.)</td>
<td>1.96</td>
<td>1.80</td>
</tr>
<tr>
<td>$N_c = 1024, n_q = 1600$</td>
<td>Error</td>
<td>=</td>
<td>=</td>
</tr>
<tr>
<td><strong>SWIFT</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m = 4$</td>
<td>Error</td>
<td>$2.70 \times 10^{-4}$</td>
<td>$1.27 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>Time (sec.)</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>$m = 5$</td>
<td>Error</td>
<td>$7.47 \times 10^{-9}$</td>
<td>$9.78 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td>Time (sec.)</td>
<td>0.01</td>
<td>0.02</td>
</tr>
<tr>
<td>$m = 6$</td>
<td>Error</td>
<td>=</td>
<td>$3.55 \times 10^{-10}$</td>
</tr>
<tr>
<td></td>
<td>Time (sec.)</td>
<td>0.02</td>
<td>0.10</td>
</tr>
<tr>
<td>$m = 7$</td>
<td>Error</td>
<td>=</td>
<td>=</td>
</tr>
<tr>
<td></td>
<td>Time (sec.)</td>
<td>0.08</td>
<td>0.34</td>
</tr>
<tr>
<td>$m = 8$</td>
<td>Error</td>
<td>=</td>
<td>=</td>
</tr>
<tr>
<td></td>
<td>Time (sec.)</td>
<td>0.33</td>
<td>1.31</td>
</tr>
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</table>

**Table:** SWIFT vs. ASCOS. Setting: GBM, $S_0 = 100$, $r = 0.0367$, $\sigma = 0.17801$, $T = 1$ and $K = 90$. The reference values are $11.9049157487$ ($N = 12$), $11.9329382045$ ($N = 50$) and $11.9405631571$ ($N = 250$).
Results on NIG

<table>
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<th>NIG</th>
<th>$N = 12$</th>
<th>$N = 50$</th>
<th>$N = 250$</th>
</tr>
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<tbody>
<tr>
<td>$N_c = 64, n_q = 100$</td>
<td>Abs error</td>
<td>$7.78 \times 10^{-3}$</td>
<td>$1.71 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>CPU time</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>$N_c = 128, n_q = 200$</td>
<td>Abs error</td>
<td>$2.60 \times 10^{-4}$</td>
<td>$5.89 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>CPU time</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>$N_c = 256, n_q = 400$</td>
<td>Abs error</td>
<td>0.19</td>
<td>0.17</td>
</tr>
<tr>
<td></td>
<td>CPU time</td>
<td>0.19</td>
<td>1.42</td>
</tr>
<tr>
<td>$N_c = 512, n_q = 800$</td>
<td>Abs error</td>
<td>1.98</td>
<td>1.96</td>
</tr>
<tr>
<td></td>
<td>CPU time</td>
<td>1.98</td>
<td>1.96</td>
</tr>
<tr>
<td>$N_c = 1024, n_q = 1600$</td>
<td>Abs error</td>
<td>14.38</td>
<td>14.22</td>
</tr>
<tr>
<td></td>
<td>CPU time</td>
<td>14.38</td>
<td>14.22</td>
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<table>
<thead>
<tr>
<th></th>
<th>SWIFT</th>
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<tbody>
<tr>
<td>$m = 4$</td>
<td>Abs error</td>
<td>$9.72 \times 10^{-2}$</td>
<td>$9.27 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>CPU time</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>$m = 5$</td>
<td>Abs error</td>
<td>$5.69 \times 10^{-3}$</td>
<td>$6.92 \times 10^{-4}$</td>
</tr>
<tr>
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<td>CPU time</td>
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<tr>
<td>$m = 6$</td>
<td>Abs error</td>
<td>$2.13 \times 10^{-4}$</td>
<td>$9.12 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>CPU time</td>
<td>0.02</td>
<td>0.12</td>
</tr>
<tr>
<td>$m = 7$</td>
<td>Abs error</td>
<td>=</td>
<td>=</td>
</tr>
<tr>
<td></td>
<td>CPU time</td>
<td>0.13</td>
<td>0.47</td>
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<tr>
<td>$m = 8$</td>
<td>Abs error</td>
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<tr>
<td></td>
<td>CPU time</td>
<td>0.39</td>
<td>1.46</td>
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Table: SWIFT vs. ASCOS. Setting: NIG, $S_0 = 100$, $r = 0.0367$, $\sigma = 0.0$, $\alpha = 6.1882$, $\beta = -3.8941$, $\delta = 0.1622$, $T = 1$ and $K = 110$. The reference values are 1.0135 ($N = 12$), 1.0377 ($N = 50$) and 1.0444 ($N = 250$).
Conclusions

- A new Fourier inversion-based technique has been proposed in the framework of discretely monitored Asian options under exponential Lévy processes.
- The application of SWIFT to the Asian pricing problem allows to overcome the main drawbacks attributed to this type of methods.
- Specially, SWIFT allows to avoid the numerical integration in the recovery of the characteristic function.
- SWIFT results in a highly accurate and fast technique, outperforming the competitors in most of the analysed situations.
Álvaro Leitao, Luis Ortiz-Gracia, and Emma I. Wagner. 
Available at SSRN: https://ssrn.com/abstract=3124902.

Luis Ortiz-Gracia and Cornelis W. Oosterlee. 
A highly efficient Shannon wavelet inverse Fourier technique for pricing European options. 

Frank Stenger.  
*Handbook of Sinc numerical methods*.  

Bowen Zhang and Cornelis W. Oosterlee. 
Efficient pricing of European-style Asian options under exponential Lévy processes based on Fourier cosine expansions. 
Acknowledgements & Questions

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More: leitao@ub.edu and alvaroleitao.github.io

Thank you for your attention
Proposition

Let \( z \in \mathbb{C} \) and \( \binom{z}{n} = \frac{z(z-1)(z-2)\cdots(z-n+1)}{n!} \). Then the series \( \sum_{n=0}^{\infty} \binom{z}{n} x^n \) converges to \( (1 + x)^z \) for all complex \( x \) with \( |x| < 1 \).

Corollary

Let \( z \in \mathbb{C} \). Then the series \( \sum_{n=0}^{\infty} \binom{z}{n} x^n y^{z-n} \) converges to \( (x + y)^z \) for all complex \( x, y \) with \( |x| < |y| \).

Proof.

The proof follows from Proposition by taking into account that \( (x + y)^z = \left( y \left[ \frac{x}{y} + 1 \right] \right)^z \).
**Proof.**

From the definition, we split the integral in two parts

\[
\hat{g}(\omega) = \int_{\mathbb{R}} e^{-i\omega x} g(x) \, dx = \int_{-\infty}^{0} e^{-i\omega x} g(x) \, dx + \int_{0}^{\infty} e^{-i\omega x} g(x) \, dx,
\]

and observe that, by Corollary above,

\[
(e^x + 1)^z = \sum_{n=0}^{\infty} \binom{z}{n} e^{nx}, \quad \text{for } x < 0, \quad \text{and} \quad (e^x + 1)^z = \sum_{n=0}^{\infty} \binom{z}{n} e^{(z-n)x}, \quad \text{for } x > 0.
\]

Replacing expressions and interchanging the integral and the sum, then we obtain,

\[
\hat{g}(\omega) = \sum_{n=0}^{\infty} \binom{z}{n} \int_{-\infty}^{0} e^{-i\omega x} e^{nx} \, dx + \sum_{n=0}^{\infty} \binom{z}{n} \int_{0}^{\infty} e^{-i\omega x} e^{(z-n)x} \, dx.
\]

Finally, solving the integrals,

\[
\hat{g}(\omega) = \sum_{n=0}^{\infty} \binom{z}{n} \frac{1}{n - i\omega} + \sum_{n=0}^{\infty} \binom{z}{n} \frac{1}{n + i(\omega + \xi)} = \sum_{n=0}^{\infty} \binom{z}{n} \frac{2n - z}{(n - i\omega)(n + i(\omega + \xi))}.
\]