Asian SWIFT method

Efficient wavelet-based valuation of arithmetic Asian options

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Motivation

- Arithmetic Asian options are still attractive in financial markets, but it numerical treatment is rather challenging.
- The valuation methods relying on Fourier inversion are highly appreciated, particularly for calibration purposes, since they are extremely fast, very accurate and easy to implement.
- Lack of robustness in the existing methods (number of terms in the expansion, numerical quadratures, truncation, etc.).
- The use of wavelets for other option problems (Europeans, early-exercise, etc.) has resulted in significant improvements in this sense.
- In the context of arithmetic Asian options, SWIFT provides extra benefits.

Outline

Problem formulation

- In Asian derivatives, the option payoff function relies on some *average* of the underlying values at a prescribed monitoring dates.
- Thus, the final value is less volatile and the option price cheaper.
- Consider $N+1$ monitoring dates $t_i \in [0, T], i = 0, \ldots, N$.
- Where T is the maturity and $\Delta t := t_{i+1} - t_i, \forall i$ (equal-spaced).
- Assume the initial state of the price process to be known, $S(0) = S_0$.
- Let averaged price be defined as $A_N := \frac{1}{N+1} \sum_{i=0}^{N} S(t_i)$, the payoff of the *European-style* Asian call option is

$$v(S, T) = (A_N - K)^+ .$$

- The risk-neutral option valuation formula,

$$v(x, t) = \mathrm{e}^{-r(T-t)} \mathbb{E}\left[v(y, T)|x\right] = \mathrm{e}^{-r(T-t)} \int_{\mathbb{R}} v(y, T) f(y|x) \mathrm{d}y,$$

with r the risk-free rate, T the maturity, $f(y|x)$ the transitional density, typically unknown, and $v(y, T)$ the payoff function.

The SWIFT method

- A structure for wavelets in $L^2(\mathbb{R})$ is called a *multi-resolution analysis*.
- We start with a family of closed nested subspaces in $L^2(\mathbb{R})$,

$$\ldots \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \ldots, \quad \bigcap_{m\in\mathbb{Z}} \mathcal{V}_m = \{0\}, \quad \overline{\bigcup_{m\in\mathbb{Z}} \mathcal{V}_m} = L^2(\mathbb{R}),$$

where

$$f(x) \in \mathcal{V}_m \Longleftrightarrow f(2x) \in \mathcal{V}_{m+1}.$$

- Then, it exists a function $\varphi \in \mathcal{V}_0$ generating an orthonormal basis, denoted by $\{\varphi_{m,k}\}_{k\in\mathbb{Z}}$, for each $\mathcal{V}_m$, $\varphi_{m,k}(x) = 2^{m/2}\varphi(2^m x - k)$.
- The function φ is called the *scaling function* or *father wavelet*.
- For any $f \in L^2(\mathbb{R})$, a projection map of $L^2(\mathbb{R})$ onto $\mathcal{V}_m$, denoted by $\mathcal{P}_m : L^2(\mathbb{R}) \to \mathcal{V}_m$, is defined by means of

$$\mathcal{P}_m f(x) = \sum_{k\in\mathbb{Z}} c_{m,k}\varphi_{m,k}(x), \quad \text{with} \quad c_{m,k} = \langle f, \varphi_{m,k} \rangle .$$

The SWIFT method

- In this work, we employ Shannon wavelets. A set of Shannon scaling functions $\varphi_{m,k}$ in the subspace $\mathcal{V}_m$ is defined as,

$$\varphi_{m,k}(x) = 2^{m/2}\frac{\sin(\pi(2^m x - k))}{\pi(2^m x - k)} = 2^{m/2}\varphi(2^m x - k), \quad k \in \mathbb{Z},$$

  where $\varphi(z) = \mathrm{sinc}(z)$, with sinc the cardinal sine function.
- Given a function $f \in L^2(\mathbb{R})$, we will consider its expansion in terms of Shannon scaling functions at the level of resolution m.
- Our aim is to recover the coefficients $c_{m,k}$ of this approximation from the Fourier transform of the function f, denoted by $\hat{f}$, defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\xi x} f(x)\mathrm{d}x,$$

  where i is the imaginary unit.

The SWIFT method

- Following wavelets theory, a function $f \in L^2(\mathbb{R})$ can be approximated at the level of resolution m by,

$$f(x) \approx \mathcal{P}_m f(x) = \sum_{k \in \mathbb{Z}} c_{m,k} \varphi_{m,k}(x),$$

where $\mathcal{P}_m f$ converges to f in $L^2(\mathbb{R})$, i.e. $\|f - \mathcal{P}_m f\|_2 \to 0$, when $m \to +\infty$.

- The infinite series is well-approximated (see Lemma 1 of [2]) by a finite summation,

$$\mathcal{P}_m f(x) \approx f_m(x) := \sum_{k=k_1}^{k_2} c_{m,k} \varphi_{m,k}(x),$$

for certain accurately chosen values k_1 and k_2.

The SWIFT method

- Computation of the coefficients $c_{m,k}$: by definition,

$$c_{m,k} = \langle f, \varphi_{m,k} \rangle = \int_{\mathbb{R}} f(x)\bar{\varphi}_{m,k}(x)\mathrm{d}x = 2^{m/2} \int_{\mathbb{R}} f(x)\varphi(2^m x - k)\mathrm{d}x.$$

- Using the classical Vieta's formula truncated with 2^{J-1} terms, the cosine product-to-sum identity and the definition of the characteristic function, the coefficients, $c_{m,k}$, can be approximated by

$$c_{m,k} \approx \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \Re \left[\hat{f}\left(\frac{(2j-1)\pi 2^m}{2^J} \right) \mathrm{e}^{\frac{\mathrm{i}k\pi(2j-1)}{2^J}} \right].$$

- Putting everything together gives the following approximation of f

$$f(x) \approx \sum_{k=k_1}^{k_2} c_{m,k}\varphi_{m,k}(x).$$

SWIFT option valuation formulas

- Truncating the integration range on $[a, b]$ and replacing density f by the SWIFT approximation,

$$v(x, t_0) \approx \mathrm{e}^{-rT} \sum_{k=k_1}^{k_2} c_{m,k} V_{m,k},$$

where,

$$V_{m,k} := \int_a^b v(y, T)\varphi_{m,k}(y|x)\mathrm{d}y.$$

- By employing the Vieta's formula again and interchanging summation and integration operations, we obtain that

$$V_{m,k} \approx \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \int_a^b v(y, T) \cos\left(\frac{2j-1}{2^J}\pi\left(2^m y - k\right)\right) \mathrm{d}y.$$

SWIFT for Asian options under exponential Lévy models

- Exponential Lévy models: $\log S(t)$, follows a Lévy process.
- The Lévy dynamics have a stationary and i.i.d. increments, fully described from its characteristic function.
- But, for arithmetic Asian options, the derivation of the corresponding characteristic function is rather involved.
- Lets start by defining the *return* or *increment* process R_i,

$$R_i := \log\left(\frac{S(t_i)}{S(t_{i-1})}\right) \quad i = 1, \ldots, N.$$

- Based on R_i, we define a new process

$$Y_i := R_{N+1-i} + Z_{i-1}, \quad i = 2, \ldots, N,$$

where $Y_1 = R_N$ and $Z_i := \log\left(1 + \mathrm{e}^{Y_i}\right), \forall i$.

SWIFT for Asian options under exponential Lévy models

- Applying the Carverhill-Clewlow-Hodges factorization to Y_i,

$$\frac{1}{N+1}\sum_{i=0}^{N} S(t_i) = \frac{\left(1+\mathrm{e}^{Y_N}\right) S_0}{N+1}.$$

- Thus, the option price for arithmetic Asian contracts can be now expressed in terms of the transitional density of the Y_N as

$$v(x,t_0) = \mathrm{e}^{-rT}\int_{\mathbb{R}} v(y,T) f_{Y_N}(y)\mathrm{d}y,$$

where $x = \log S_0$ and the call payoff function is given by

$$v(y,T) = \left(\frac{S_0\left(1+\mathrm{e}^{y}\right)}{N+1} - K\right)^{+}$$

- Again, the probability density function f_{Y_N} is generally not known, even for Lévy processes. However, as the process Y_N is defined in a recursive manner, the characteristic function of Y_N can be computed iteratively as well.

Characteristic function of Y_N

- By the definition of Y_i, the initial and recursive characteristic functions are

$$\hat{f}_{Y_1}(\xi) = \hat{f}_{R_N}(\xi) = \hat{f}_R(\xi),$$
$$\hat{f}_{Y_i}(\xi) = \hat{f}_{R_{N+1-i}+Z_{i-1}}(\xi) = \hat{f}_{R_{N+1-i}}(\xi) \cdot \hat{f}_{Z_{i-1}}(\xi) = \hat{f}_R(\xi) \cdot \hat{f}_{Z_{i-1}}(\xi).$$

- By definition, the characteristic function of Z_{i-1} reads

$$\hat{f}_{Z_{i-1}}(\xi) := \mathbb{E}\left[\mathrm{e}^{-\mathrm{i}\xi \log\left(1+\mathrm{e}^{Y_{i-1}}\right)}\right] = \int_{\mathbb{R}} (1+\mathrm{e}^x)^{-\mathrm{i}\xi} f_{Y_{i-1}}(x)\mathrm{d}x.$$

- We can again apply the wavelet approximation to $f_{Y_{i-1}}$ as

$$\hat{f}_{Z_{i-1}}(\xi) \approx \int_{\mathbb{R}} (1+\mathrm{e}^x)^{-\mathrm{i}\xi} \sum_{k=k_1}^{k_2} c_{m,k}\varphi_{m,k}(x)\mathrm{d}x$$
$$= 2^{\frac{m}{2}} \sum_{k=k_1}^{k_2} c_{m,k} \int_{\mathbb{R}} (\mathrm{e}^x+1)^{-\mathrm{i}\xi} \operatorname{sinc}(2^m x - k)\,\mathrm{d}x.$$

Characteristic function of Y_N

- The integral on the right hand side needs to be computed efficiently to make the method easily implementable, robust and very fast.
- State-of-the-art methods from the literature rely on solving the integral by means of quadratures.

Theorem (Theorem 1.3.2 of [3])

Let f be defined on $\mathbb{R}$ and let its Fourier transform $\hat{f}$ be such that for some positive constant d, $|\hat{f}(\omega)| = \mathcal{O}\left(\mathrm{e}^{-d|\omega|}\right)$ for $\omega \to \pm\infty$, then as $h \to 0$

$$\frac{1}{h}\int_{\mathbb{R}} f(x) S_{j,h}(x)\mathrm{d}x - f(jh) = \mathcal{O}\left(\mathrm{e}^{-\frac{\pi d}{h}}\right),$$

where $S_{j,h}(x) = \operatorname{sinc}\left(\frac{x}{h} - j\right)$ for $j \in \mathbb{Z}$.

- Theorem 1 allows us to approximate the integral above provided that $g(x) := (\mathrm{e}^x + 1)^{-\mathrm{i}\xi}$ satisfies the hypothesis.

Characteristic function of Y_N

- If we consider $h = \frac{1}{2^m}$, then it follows from Theorem 1 that

$$\int_{\mathbb{R}} g(x)\mathrm{sinc}\left(2^m x - k\right)\mathrm{d}x \approx hg\left(kh\right) = \frac{1}{2^m}\left(\mathrm{e}^{\frac{k}{2^m}} + 1\right)^{-\mathrm{i}\xi}.$$

- Thus, $\hat{f}_{Z_{i-1}}$ can be approximated by

$$\hat{f}_{Z_{i-1}}(\xi) \approx 2^{-\frac{m}{2}} \sum_{k=k_1}^{k_2} c_{m,k}\left(\mathrm{e}^{\frac{k}{2^m}} + 1\right)^{-\mathrm{i}\xi}.$$

- Finally,

$$\hat{f}_{Y_i}(\xi) = \hat{f}_R(\xi)\hat{f}_{Z_{i-1}} \approx \hat{f}_R(\xi)2^{-\frac{m}{2}} \sum_{k=k_1}^{k_2} c_{m,k}\left(\mathrm{e}^{\frac{k}{2^m}} + 1\right)^{-\mathrm{i}\xi},$$

where the density coefficients $c_{m,k}$ are computed as follows

$$c_{m,k} \approx \frac{2^{m/2}}{2^{J-1}} \sum_{j=0}^{2^{J-1}} \Re\left\{\hat{f}_{Y_{i-1}}\left(\frac{(2j-1)\pi 2^m}{2^J}\right)\mathrm{e}^{\frac{\mathrm{i}k\pi(2j-1)}{2^J}}\right\}.$$

- It remains to prove that function $g(x) = (\mathrm{e}^x + 1)^{-\mathrm{i}\xi}$ satisfies $|\hat{g}(\omega)| = \mathcal{O}\left(\mathrm{e}^{-d|\omega|}\right)$ for $\omega \to \pm\infty$.
- We have derived an expression for $\hat{g}(w)$,

Proposition

Let $g(x) = (\mathrm{e}^x + 1)^z$, where $z = -\mathrm{i}\xi$ and $x, \xi \in \mathbb{R}$. Then,

$$\hat{g}(\omega) = \sum_{n=0}^{\infty} \binom{z}{n} \frac{2n - z}{(n - \mathrm{i}\omega)(n + \mathrm{i}(\omega + \xi))}, \quad \omega \in \mathbb{R}.$$

- It is rather complicated to get a closed-form solution for the modulus of $\hat{g}(\omega)$ from this expression.

Characteristic function of Y_N

- By employing Wolfram Mathematica 11.2, the infinite sum is written as

$$\hat{g}(\omega) = \frac{\xi}{2\omega + \xi} \left[\mathrm{e}^{-\pi\omega} \left(B_{-1}\left(-\mathrm{i}\omega, 1+z\right) + 2B_{-1}\left(1-\mathrm{i}\omega, z\right)\right) + \right.$$
$$+ \Gamma\left(\mathrm{i}\omega - z\right) \left(2(\mathrm{i}\omega - z)\, {}_2\tilde{F}_1\left(1-z, 1+\mathrm{i}\omega - z; 2+\mathrm{i}\omega - z; -1\right) - \right.$$
$$\left.\left. - \ {}_2\tilde{F}_1\left(-z, \mathrm{i}\omega - z; 1+\mathrm{i}\omega - z; -1\right)\right)\right],$$

in terms of gamma, Γ, beta, B, and regularized hypergeometric, ${}_2\tilde{F}_1(a, b; c; \nu)$.

Characteristic function of Y_N

- Representing $|\hat{g}(\omega)|$,

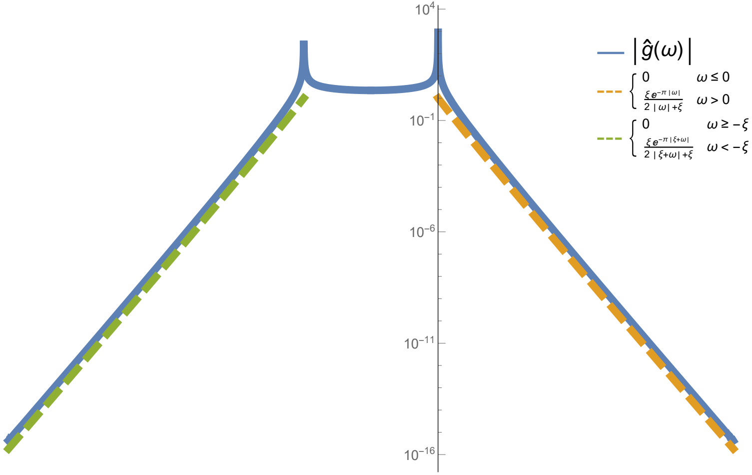

Figure: Modulus of $\hat{g}(\omega)$.

Payoff coefficients

- To complete the SWIFT pricing formula, compute the payoff coefficients, $V_{m,k}$,

$$V_{m,k} = \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \left[\frac{S_0}{N+1} \left(I_2^{j,k}(\tilde{x}, b) + I_0^{j,k}(\tilde{x}, b) \right) - K I_0^{j,k}(\tilde{x}, b) \right],$$

where $\tilde{x} = \log\left(\frac{K(N+1)}{S_0} - 1\right)$ and the functions $I_0^{j,k}$ and $I_2^{j,k}$ are defined by the following integrals

$$I_0^{j,k}(x_1, x_2) := \int_{x_1}^{x_2} \cos\left(C_j\left(2^m y - k\right)\right) \mathrm{d}y,$$

$$I_2^{j,k}(x_1, x_2) := \int_{x_1}^{x_2} \mathrm{e}^y \cos\left(C_j\left(2^m y - k\right)\right) \mathrm{d}y,$$

with $C_j = \frac{2j-1}{2^J}\pi$. These integrals are analytically available.

Numerical results

- We compare the SWIFT method against a state-of-the-art method, the well-known COS method, particularly the COS variant for arithmetic Asian option, called *ASCOS* method [4].
- To the best of our knowledge, the ASCOS method provides the best balance between accuracy and efficiency.
- Arithmetic Asian call option valuation with varying number of monitoring dates, $N = 12$ (monthly), $N = 50$ (weekly) and $N = 250$ (daily), and conceptually different underlying Lévy dynamics: *Geometric Brownian motion* (GBM) and *Normal inverse Gaussian* (NIG).
- We assess not only the accuracy in the solution but also the computational performance.
- All the experiments have been conducted in a computer system with the following characteristics: CPU Intel Core i7-4720HQ 2.6GHz and memory of 16GB RAM. The employed software package is Matlab R2017b.

Results on GBM

GBM		$N=12$	$N=50$	$N=250$
		ASCOS		
$N_c=64,\ n_q=100$	Error Time (sec.)	3.75×10^{-4} 0.03	8.34×10^{-4} 0.02	7.17×10^{-3} 0.01
$N_c=128,\ n_q=200$	Error Time (sec.)	8.37×10^{-7} 0.03	7.43×10^{-6} 0.02	3.82×10^{-5} 0.02
$N_c=256,\ n_q=400$	Error Time (sec.)	= 0.16	5.33×10^{-7} 0.12	1.58×10^{-7} 0.11
$N_c=512,\ n_q=800$	Error Time (sec.)	= 1.96	= 1.80	3.04×10^{-8} 1.85
$N_c=1024,\ n_q=1600$	Error Time (sec.)	= 13.99	= 13.99	= 14.25
		SWIFT		
$m=4$	Error Time (sec.)	2.70×10^{-4} 0.01	1.27×10^{-2} 0.01	3.82×10^{-2} 0.03
$m=5$	Error Time (sec.)	7.47×10^{-9} 0.01	9.78×10^{-5} 0.02	4.01×10^{-3} 0.06
$m=6$	Error Time (sec.)	= 0.02	3.55×10^{-10} 0.10	6.96×10^{-4} 0.40
$m=7$	Error Time (sec.)	= 0.08	= 0.34	1.21×10^{-8} 1.37
$m=8$	Error Time (sec.)	= 0.33	= 1.31	= 5.11

Table: SWIFT vs. ASCOS. Setting: **GBM**, $S_0=100$, $r=0.0367$, $\sigma=0.17801$, $T=1$ and $K=90$. The reference values are 11.9049157487 ($N=12$), 11.9329382045 ($N=50$) and 11.9405631571 ($N=250$).

Results on NIG

NIG		$N = 12$	$N = 50$	$N = 250$
		ASCOS		
$N_c = 64,\ n_q = 100$	Abs error CPU time	7.78×10^{-3} 0.03	1.71×10^{-1} 0.03	8.75×10^{-2} 0.02
$N_c = 128,\ n_q = 200$	Abs error CPU time	2.60×10^{-4} 0.03	5.89×10^{-3} 0.03	1.49×10^{-2} 0.03
$N_c = 256,\ n_q = 400$	Abs error CPU time	= 0.19	= 0.17	1.42×10^{-4} 0.15
$N_c = 512,\ n_q = 800$	Abs error CPU time	= 1.98	= 1.96	= 2.02
$N_c = 1024,\ n_q = 1600$	Abs error CPU time	= 14.38	= 14.22	= 14.71
		SWIFT		
$m = 4$	Abs error CPU time	9.72×10^{-2} 0.02	9.27×10^{-2} 0.02	4.01×10^{-2} 0.04
$m = 5$	Abs error CPU time	5.69×10^{-3} 0.02	6.92×10^{-4} 0.03	4.50×10^{-3} 0.08
$m = 6$	Abs error CPU time	2.13×10^{-4} 0.02	9.12×10^{-4} 0.12	9.11×10^{-4} 0.48
$m = 7$	Abs error CPU time	= 0.13	= 0.47	= 1.52
$m = 8$	Abs error CPU time	= 0.39	= 1.46	= 5.85

Table: SWIFT vs. ASCOS. Setting: **NIG**, $S_0 = 100$, $r = 0.0367$, $\sigma = 0.0$, $\alpha = 6.1882$, $\beta = -3.8941$, $\delta = 0.1622$, $T = 1$ and $K = 110$. The reference values are 1.0135 ($N = 12$), 1.0377 ($N = 50$) and 1.0444 ($N = 250$).

Conclusions

- A new Fourier inversion-based technique has been proposed in the framework of discretely monitored Asian options under exponential Lévy processes.
- The application of SWIFT to the Asian pricing problem allows to overcome the main drawbacks attributed to this type of methods.
- Specially, SWIFT allows to avoid the numerical integration in the recovery of the characteristic function.
- SWIFT results in a highly accurate and fast technique, outperforming the competitors in most of the analysed situations.

References

Álvaro Leitao, Luis Ortiz-Gracia, and Emma I. Wagner.
SWIFT valuation of discretely monitored arithmetic Asian options, 2018.
Available at SSRN: https://ssrn.com/abstract=3124902.

Luis Ortiz-Gracia and Cornelis W. Oosterlee.
A highly efficient Shannon wavelet inverse Fourier technique for pricing European options.
SIAM Journal on Scientific Computing, 38(1):118–143, 2016.

Frank Stenger.
Handbook of Sinc numerical methods.
CRC Press, Inc., Boca Raton, FL, USA, 2010.

Bowen Zhang and Cornelis W. Oosterlee.
Efficient pricing of European-style Asian options under exponential Lévy processes based on Fourier cosine expansions.
SIAM Journal on Financial Mathematics, 4(1):399–426, 2013.

Acknowledgements & Questions

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Thank you for your attention

Bonus - Proof expression $\hat{g}(\omega)$

Proposition

Let $z \in \mathbb{C}$ *and* $\binom{z}{n} = \frac{z(z-1)(z-2)\cdots(z-n+1)}{n!}$. *Then the series* $\sum_{n=0}^{\infty} \binom{z}{n} x^n$ *converges to* $(1+x)^z$ *for all complex* x *with* $|x| < 1$.

Corollary

Let $z \in \mathbb{C}$. *Then the series* $\sum_{n=0}^{\infty} \binom{z}{n} x^n y^{z-n}$ *converges to* $(x+y)^z$ *for all complex* x, y *with* $|x| < |y|$.

Proof.

The proof follows from Proposition by taking into account that $(x+y)^z = \left(y\left[\frac{x}{y}+1\right]\right)^z$. □

Bonus - Proof expression $\hat{g}(\omega)$

Proof.

From the definition, we split the integral in two parts

$$\hat{g}(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega x} g(x)\mathrm{d}x = \int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}\omega x} g(x)dx + \int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}\omega x} g(x)\mathrm{d}x,$$

and observe that, by Corollary above,

$$(\mathrm{e}^{x}+1)^{z} = \sum_{n=0}^{\infty} \binom{z}{n} \mathrm{e}^{nx}, \quad \text{for } x<0, \quad \text{and} \quad (\mathrm{e}^{x}+1)^{z} = \sum_{n=0}^{\infty} \binom{z}{n} \mathrm{e}^{(z-n)x}, \quad \text{for } x>0.$$

Replacing expressions and interchanging the integral and the sum, then we obtain,

$$\hat{g}(\omega) = \sum_{n=0}^{\infty} \binom{z}{n} \int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}\omega x} \mathrm{e}^{nx} \mathrm{d}x + \sum_{n=0}^{\infty} \binom{z}{n} \int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}\omega x} \mathrm{e}^{(z-n)x} \mathrm{d}x.$$

Finally, solving the integrals,

$$\hat{g}(\omega) = \sum_{n=0}^{\infty} \binom{z}{n} \frac{1}{n-\mathrm{i}\omega} + \sum_{n=0}^{\infty} \binom{z}{n} \frac{1}{n+\mathrm{i}(\omega+\xi)} = \sum_{n=0}^{\infty} \binom{z}{n} \frac{2n-z}{(n-\mathrm{i}\omega)(n+\mathrm{i}(\omega+\xi))}.$$