III Congreso XoveTIC Talento científico

Machine learning to compute implied volatility from European/American options considering dividend yield

1. Problem formulation

The Black-Scholes model for pricing European options reads,

$$\frac{\partial V_{eu}}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V_{eu}}{\partial S^2}+(r-q)S\frac{\partial V_{eu}}{\partial S}-rV_{eu}=0, \tag{1}$$

where r and q are the risk-free interest rate and continuous dividend yield, respectively. For American options, the original Black-Scholes equation becomes a variational inequality,

$$\frac{\partial V_{am}}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V_{am}}{\partial S^2}+(r-q)S\frac{\partial V_{am}}{\partial S}-rV_{am}\leq 0, \tag{2}$$

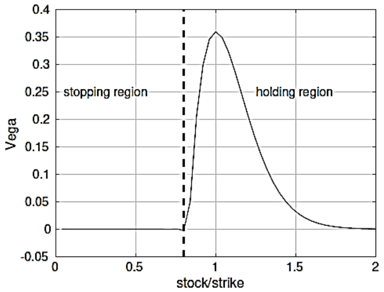
where the free boundary condition is $V_{am}(S,t)\geq H(K,S_t)$, and the terminal condition is $V_{am}(S,T)\geq H(K,S_T)$. The European/American Black-Scholes solution is denoted by $V_{eu/am}=BS_{eu/am}(\sigma,S_0,K,\tau,r,q,\alpha)$. Given an observed market option price, V^{mkt} (European or American), the Black-Scholes implied volatility, $\sigma^*$is defined by

$$BS(\sigma^*;S_0,K,\tau,r,q,\alpha)=V^{mkt}. \tag{3}$$

There does not exist a closed-form expression of the inverse function for neither European-style nor American-style options. A popular way is to formulate the above problem into a minimization problem,

$$\min_{\sigma^*\in R^+} BS(\sigma^*;S_0,K,\tau,r,q,\alpha)-V^{mkt} \tag{4}$$

However, some issues are likely to arise when using derivative-based root-finding numerical algorithms to solve (4), see figure below.



2. Methodology

In order to avoid an iterative algorithm, we provide a data-driven approach for directly approximating the inverse function of (3) via neural networks. Mathematically, an *artificial neural network* (ANN) can be represented as a composite function,

$$\mathrm{F}(\mathbf{x}|\mathbf{\Theta})=f^{(L)}(\ldots f^{(2)}(f^{(1)}(\mathbf{x};\theta_1);\theta_2);\ldots\theta_L), \tag{6}$$

where $\boldsymbol{x}$ stands for the input variables, $\mathbf{\Theta}$ for the hidden parameters (i.e. weights and biases) and L for the number of hidden layers.

The implied volatility defined by Equation (3) can be written as an inverse function of the pricing model,

$$\sigma^*=BS^{-1}(V^{mkt};S,K,\tau,r,q,\alpha), \tag{7}$$

where $BS^{-1}(\cdot)$ denotes the inverse Black-Scholes function (European-style or American-style). We use a deep neural network to approximate the inverse Black-Scholes function,

$$\sigma^*=BS^{-1}(V^{mkt};S,K,\tau,r,q,\alpha)\approx \mathrm{NN}(V^{mkt};S,K,\tau,r,q,\alpha). \tag{8}$$

2.1 ANN for European implied volatility

The inverse Black-Scholes function probably gives rise to steep gradients of the volatility with respect to the option price. It is known that the ANN has difficulties accurately representing such gradients. Here we employ the gradient-squashing technique to address this issue,

$$\hat{V}^P_{eu}=\log\left(V^P_{eu}(S,t)-\max(Ke^{-r\tau}-S_te^{-q\tau},0)\right). \tag{9}$$

2.2 ANN for American implied volatility

For training the ANN to compute implied volatility from American options, there are two steps, due to the early-exercise feature. First, we need to compute again the gradient-squashed time value of an American option,

$$\hat{V}^P_{am}=\log\left(V^P_{am}(S_t,t)-\max(K-S_t,Ke^{-r\tau}-S_te^{-q\tau},0)\right). \tag{10}$$

Second, the effective definition domain Ω_h of the inverse function (8) is numerically found based on the generated samples.

3. Numerical results

Table 1. ANN hyper-parameters.

Parameters	Values
Hidden layers	4
Neurons(each layer)	400
Activation	ReLU
Initialization	Glorot
Optimizer	Adam
Batch size	1024

Table 2. Model parameter ranges.

	Parameters	Range
Inputs	Stock price (K/S_0)	[0.3, 1.8]
	Time to maturity (τ)	[0.08, 2.5]
	Risk-free rate (r)	[0.0, 0.25]
	Dividend yield (q)	[0.0, 0.25]
	Scaled time value ($\hat{V}$)	-
Output	Volatility (σ)	(0.01, 1.05)

Table 3. Performance of IV-ANN.

Phase	European options				American options			
	MSE	MAE	MAPE	R^2	MSE	MAE	MAPE	R^2
Training	$1.72\cdot10^{-7}$	$3.17\cdot10^{-4}$	$6.99\cdot10^{-4}$	0.9999976	$7.12\cdot10^{-7}$	$5.66\cdot10^{-4}$	$1.42\cdot10^{-4}$	0.999990
Testing	$1.94\cdot10^{-7}$	$3.35\cdot10^{-4}$	$7.39\cdot10^{-4}$	0.9999972	$1.93\cdot10^{-6}$	$6.52\cdot10^{-4}$	$2.35\cdot10^{-3}$	0.999974

Table 4. Computational cost based on 20000 option prices.

Method	GPU (seconds)	CPU (seconds)	Robustness
Newton-Raphson	19.68	23.06	No
Brent	52.08	60.67	Yes
Bi-section	337.94	390.91	Yes
IV-ANN	0.20	1.90	Yes

Besides of being robust, the neural network (IV-ANN) solver is much faster than an iterative numerical solver to compute implied volatility from European/American options including dividends.

References

Liu, S.; Leitao, A.; Borovykh, A.; Oosterlee, C.W. On Calibration Neural Networks for extracting implied information from American options, 2020. Available at arXiv: https://arxiv.org/abs/2001.11786.

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