A stochastic $\theta$-SEIHRD model
Adding randomness to the COVID-19 spread

Álvaro Leitao and Carlos Vázquez

alvaro.leitao@udc.gal
https://alvaroleitao.github.io/

December 18, 2020
Motivation

- Classical compartmental SEIR-like models are too simplistic.
- Particular COVID-19 characteristics: undetected, hospitalized, deaths, etc.
- Require a COVID-19 ad-hoc model: $\theta$-SEIHRD model.
- Deterministic version: rigid and limited information.
- Uncertainty may influence the compartments dynamics.
- Behavioural effects, public interventions, seasonal patterns, environmental factors, etc. are factors with a random component.
- How to account for it? Stochastic extension!
Outline

1. Introduction
2. A stochastic compartmental model for the COVID-19
   - Original model
   - Stochastic extension
3. Numerical solution of the stochastic $\theta$-SEIHRD model
4. Numerical and statistical analysis
5. Conclusions
The compartmental models are formulated in a deterministic fashion: *Ordinary Differential Equations* (ODEs).

There are two common approaches to include stochasticity into a deterministic model:

- *Continuous Time Markov Chain* (CTMC).
- *Stochastic Differential Equations* (SDEs).

The stochastic models allow to capture many kinds of circumstances including uncertainty.

The solution of the stochastic model is a set of stochastic processes, containing much more information than the deterministic analogous.

Statistical analyses can be performed (expectations, quantiles or worst case scenarios).
From deterministic to stochastic

- We follow the SDE approach incorporating a *Brownian motion* to the ODEs.

- Two common ways of addressing this kind of stochastic extension:
  - Adding arbitrary random noise.
  - Perturbing one (or more) of the existing model parameters.

- We choose the second alternative for interpretability purposes.

- In practice, the uncertainty will have impact on a particular model component, typically represented by a model parameter.

- A randomly perturbed parameter can be reasonably explained in terms of the variability produced by the source of the considered uncertainty.
We consider the (simplified) $\theta$-SEIHRD model from [2].

Consisting in 9 equations, 6 coupled,

\[
\frac{dS}{dt}(t) = -\frac{S(t)}{N} \left( m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_u}(t)\beta_{I_u}(\theta(t))I_u(t) \right)
\]
\[
- \frac{S(t)}{N} \left( m_{H_R}(t)\beta_{H_R}(t)H_R(t) + m_{H_D}(t)\beta_{H_D}(t)H_D(t) \right),
\]
\[
\frac{dE}{dt}(t) = \frac{S(t)}{N} \left( m_E(t)\beta_E E(t) + m_I(t)\beta_I I(t) + m_{I_u}(t)\beta_{I_u}(\theta(t))I_u(t) \right)
\]
\[
+ \frac{S(t)}{N} \left( m_{H_R}(t)\beta_{H_R}(t)H_R(t) + m_{H_D}(t)\beta_{H_D}(t)H_D(t) \right) - \gamma_E E(t),
\]
\[
\frac{dl}{dt}(t) = \gamma_E E(t) - \gamma_I(t)l(t),
\]
\[
\frac{dl_u}{dt}(t) = (1 - \theta(t))\gamma_I(t)l(t) - \gamma_{I_u}(t)I_u(t),
\]
\[
\frac{dH_R}{dt}(t) = \theta(t) \left( 1 - \frac{\omega(t)}{\theta(t)} \right) \gamma_I(t)l(t) - \gamma_{H_R}(t)H_R(t),
\]
\[
\frac{dH_D}{dt}(t) = \omega(t)\gamma_I(t)l(t) - \gamma_{H_D}(t)H_D(t),
\]
and 3 uncoupled equations,

\[
\frac{dR_d}{dt}(t) = \gamma_{HR}(t)H_R(t),
\]

\[
\frac{dR_u}{dt}(t) = \gamma_{lu}(t)I_u(t),
\]

\[
\frac{dD}{dt}(t) = \gamma_{HD}(t)H_D(t).
\]

whose solution can be obtained by

\[
R_d(t) = R_d(t_0) + \int_{t_0}^{t} \gamma_{HR}(s)H_R(s)\,ds,
\]

\[
R_u(t) = R_u(t_0) + \int_{t_0}^{t} \gamma_{lu}(s)I_u(s)\,ds,
\]

\[
D(t) = D(t_0) + \int_{t_0}^{t} \gamma_{HD}(s)H_D(s)\,ds.
\]
Figure: The $\theta$-SEIHRD model diagram.
Efficiency of the control measures

\[ m_E, m_I, m_{Iu}, m_{HR}, m_{HD} \in [0, 1](\%) \]. Here, only one control measure is assumed, implemented at date \( \lambda_1 \),

\[
m_X(t) = \begin{cases} 
1, & \text{if } t \in [0, \lambda_1], \\
\exp(-\kappa_1(t - \lambda_1)), & \text{if } t \in [\lambda_1, T],
\end{cases}
\]

with the parameter \( \kappa_1 \in [0, 0.2] \).

The fatality rate \( \omega(t) \in [\underline{\omega}, \overline{\omega}] \subset [0, 1] \),

\[
\omega(t) = m_I(t) \overline{\omega} + (1 - m_I(t)) \underline{\omega},
\]

with \( \underline{\omega} \) and \( \overline{\omega} \) being the fatality rate limits with and without control measures, \( \overline{\omega} = \omega + \delta_\omega \).
The fraction of detected individuals, $\theta \in [\overline{\theta}, 1]$, 

$$\theta(t) = \begin{cases} 
\theta, & \text{if } t \in [t, \lambda_1], \\
\text{linear continuous}, & \text{if } t \in [\lambda_1, \lambda_2], \\
\overline{\theta}, & \text{if } t \in [\lambda_2, T], 
\end{cases}$$

with $\theta, \overline{\theta}, \lambda_1, \lambda_2$ inferred from the data.

Compartment transition rates $\gamma_E, \gamma_I, \gamma_{Iu}, \gamma_{HR}, \gamma_{HD} \in (0, +\infty)$. Given the days in each compartment, $d_E, d_I, d_{Iu}, d_{HR}$ and $d_{HD}$, with $d_{Iu} = d_{HR}$ and $d_{HD} = d_{HR} + \delta_R, \delta_R > 0$,

$$\gamma_I = \frac{1}{d_E}, \quad \gamma_{Iu}(t) = \gamma_{HR}(t) = \frac{1}{d_{Iu} + g(t)}$$

$$\gamma_I(t) = \frac{1}{d_I - g(t)}, \quad \gamma_{HD}(t) = \frac{1}{d_{Iu} + g(t) + \delta_R},$$

where $g(t) = d_g(1 - m_I(t))$. 
The disease contact rates $\beta_E, \beta_I, \beta_1, \beta_{HR}, \beta_{HD} \in \mathbb{R}^+$. The parameter $\beta_I$ is calibrated. More $\beta_E = C_E \beta_I$, $\beta_{HR} = \beta_{HD} = C_H(t) \beta_I$ and

$$\beta_{1u}(t) = \beta_I + \frac{\beta_I - \beta_{1}}{1 - \omega(t)} (1 - \theta(t)),$$

where $\beta_{1} = C_u \beta_I$, with $C_E$, $C_H(t)$ and $C_u \in [0, 1]$. Parameters $C_E$ and $C_u$ are obtained calibration, while

$$C_H(t) = \frac{\alpha_H \left( \frac{\beta_I}{\gamma_I(t)} + \frac{\beta_E}{\gamma_E(t)} + (1 - \theta(t)) \frac{\beta_{1u}(t)}{\gamma_{1u}(t)} \right)}{(1 - \alpha_H) \beta_I \theta(t) \left( \left(1 - \frac{\omega(t)}{\theta(t)} \right) \frac{1}{\gamma_{HR}(t)} + \frac{\omega(t)}{\theta(t)} \frac{1}{\gamma_{HR}(t)} \right)}$$

with $\alpha_H$ being the percentage of healthcare workers infected.
Our aim is to introduce stochasticity to the simplified $\theta$-SEIHRD model.

We add randomness on the disease contact rates, $\beta$’s.

Writing them in terms of $\beta_I$,

$$
\begin{align*}
\beta_E &= \beta_I A_E, \\
\beta_{Iu} &= \beta_I A_{Iu}, \\
\beta_{HR} &= \beta_I A_{HR}, \\
\beta_{HD} &= \beta_I A_{HD},
\end{align*}
$$

where

$$
\begin{align*}
A_E &= C_E, \\
A_{Iu}(t) &= C_u + \frac{(1 - C_u)(1 - \theta(t))}{1 - \omega(t)}, \\
A_{HR}(t) &= A_{HD}(t) = \frac{\alpha_H \left( \frac{1}{\gamma_I(t)} + \frac{A_E}{\gamma_E} + (1 - \theta(t)) \frac{A_{Iu}(t)}{\gamma_{Iu}(t)} \right)}{(1 - \alpha_H)\theta(t) \left( (1 - \frac{\omega(t)}{\theta(t)}) \frac{1}{\gamma_{HR}(t)} + \frac{\omega(t)}{\theta(t)} \frac{1}{\gamma_{HD}(t)} \right)}.
\end{align*}
$$
The $\theta$-SEIHRD model can be rewritten as,

\[
\frac{dS}{dt}(t) = -\beta_I \frac{S(t)M(t)}{N},
\]

\[
\frac{dE}{dt}(t) = \beta_I \frac{S(t)M(t)}{N} - \gamma_E E(t),
\]

\[
\frac{dI}{dt}(t) = \gamma_E E(t) - \gamma_I I(t),
\]

\[
\frac{dl_u}{dt}(t) = (1 - \theta(t))\gamma_I I(t) - \gamma_{l_u} l_u(t),
\]

\[
\frac{dH_R}{dt}(t) = \theta(t) \left( 1 - \frac{\omega(t)}{\theta(t)} \right) \gamma_I I(t) - \gamma_{H_R} H_R(t),
\]

\[
\frac{dH_D}{dt}(t) = \omega(t)\gamma_I I(t) - \gamma_{H_D} H_D(t),
\]

where

\[
M(t) = m_E A_E E(t) + m_I I(t) + m_{l_u} A_{l_u} l_u(t)
\]

\[+ m_{H_R} A_{H_R} H_R(t) + m_{H_D} A_{H_D} H_D(t).\]
Replace the constant parameter $\beta_I$ by a random walk.

The disease contact rate in compartment $I$ follows a newly introduced stochastic process, $\tilde{\beta}_I(t)$.

We choose the well-known CIR process [1].

The main advantage of the CIR process: it ensures the spacial states to be non-negative.

Further, the CIR process is a mean-reverting process.

The dynamics of $\tilde{\beta}_I$ read

$$d\tilde{\beta}_I(t) = \nu_{\beta_I}(\mu_{\beta_I} - \tilde{\beta}_I(t))dt + \sigma_{\beta_I}\sqrt{\tilde{\beta}(t)}dW(t)$$

where $\nu_{\beta_I}$ is the mean reverting speed, $\mu_{\beta_I}$ is the long-term average, $\sigma_{\beta_I}$ is the volatility and $dW(t)$ is a Brownian motion increment.
The system of SDEs governing the stochastic $\theta$-SEIHRD model is given by,

$$
\begin{align*}
    dS(t) &= \tilde{\beta}_I(t) \frac{S(t)M(t)}{N} dt \\
    dE(t) &= \left( \tilde{\beta}_I(t) \frac{S(t)M(t)}{N} - \gamma_E E(t) \right) dt \\
    dl(t) &= \left( \gamma_E E(t) - \gamma_I I(t) \right) dt, \\
    dl_u(t) &= \left( (1 - \theta(t)) \gamma_I I(t) - \gamma_I u I_u(t) \right) dt, \\
    dH_R(t) &= \left( \theta(t) \left( 1 - \frac{\omega(t)}{\theta(t)} \right) \gamma_I I(t) - \gamma_{HR} H_R(t) \right) dt, \\
    dH_D(t) &= \left( \omega(t) \gamma_I I(t) - \gamma_{HD} H_D(t) \right) dt, \\
    dR_d(t) &= \gamma_{HR}(t) H_R(t) dt, \\
    dR_u(t) &= \gamma_{Iu}(t) I_u(t) dt, \\
    dD(t) &= \gamma_{HD}(t) H_D(t) dt, \\
    d\tilde{\beta}_I(t) &= \nu_{\beta_1} (\mu_{\beta_1} - \tilde{\beta}_I(t)) dt + \sigma_{\beta_1} \sqrt{\tilde{\beta}_I(t)} dW(t).
\end{align*}
$$

with $M(t)$ as defined before.
Although only one source of randomness is introduced, the solution of the whole system becomes a set of stochastic processes.

The remaining equations depend on the first equations for $S$ and $E$, which present a dependence on $\tilde{\beta}_I$.

The CIR process is widely employed to simulate the evolution of interest rates in quantitative finance.

In some sense, the interest rates in finance and the disease contact rates in epidemiology present a rather similar behaviour: positiveness, controlled variability and long-term stability.
For a set of constant initial data $S(0), E(0), I(0), I_u(0), H_R(0), H_D(0), R_d(0), R_u(0), D(0)$ and $\tilde{\beta}_I(0)$, the system of SDEs has a unique strong solution.

As the system of SDEs is nonlinear, it is not possible to obtain a closed-form expression for the solution.

The use of numerical methods becomes mandatory. We adopt the following strategy:

1. Perform a simulation of the dynamics of $\tilde{\beta}_I(t)$, in accordance with the CIR process.
2. Solve the resulting ODE system for each path of $\tilde{\beta}_I(t)$.

We obtain a set of random walks for each stochastic process representing a model variable.
The CIR process is a well-studied dynamics often employed in computational finance.

The underlying distribution is known analytically, relying on the non-central chi-squared distribution.

Given two time points, $s$ and $t$, $s < t$, the conditional distribution of $\tilde{\beta}_I$ reads

$$\tilde{\beta}_I(t) | \tilde{\beta}_I(s) \sim c(t, s) \cdot \chi^2 \left( d, \frac{e^{-\nu \beta_I(t-s)}}{c(t, s)} \tilde{\beta}_I(s) \right),$$

where

$$c(t, s) = \frac{\sigma_{\beta_I}^2}{4 \mu_{\beta_I}} \left( 1 - e^{-\nu \beta_I(t-s)} \right), \quad d = \frac{4 \nu \beta_I \mu_{\beta_I}}{\sigma_{\beta_I}^2},$$

and $\chi^2(a, b)$ is the non-central chi-squared distribution with $a$ degrees of freedom and non-centrality parameter $b$. 
We can then define an exact simulation scheme, which can be used to obtain realizations of $\tilde{\beta}_I$.

Given a set of $m + 1$ time points, $\{t_i\}_{0}^{m}$, where the solution will be computed, we have, for $i = 0, \ldots, m - 1$,

$$c(t_{i+1}, t_i) = \frac{\sigma_{\beta_I}^2}{4\mu_{\beta_I}} \left( 1 - e^{-\nu_{\beta_I}(t_{i+1} - t_i)} \right),$$

$$\tilde{\beta}_I(t_{i+1}) = c(t_{i+1}, t_i) \chi^2 \left( d, \frac{e^{-\nu_{\beta_I}(t_{i+1} - t_i)}}{c(t_{i+1} - t_i)} \tilde{\beta}_I(t_i) \right)$$

given some initial value $\tilde{\beta}_I(t_0) = \tilde{\beta}_I(0)$.

By employing this scheme, we generate $n$ simulated discrete sample paths of $\tilde{\beta}_I$. 
The stochastic $\theta$-SEIHRD model
Potential

**Figure:** Deterministic vs. Stochastic: $\nu_{\beta_I} = 1$, $\mu_{\beta_I} = \beta_I$ and $\sigma_{\beta_I} = 0.1$, with $n = 2^{15}$ Monte Carlo simulations (only 8 simulations depicted).
The stochastic $\theta$-SEIHRD model

Potential

Figure: Deterministic vs. Stochastic: $\nu_{\beta_I} = 1$, $\mu_{\beta_I} = \beta_I$ and $\sigma_{\beta_I} = 0.1$, with $n = 2^{15}$ Monte Carlo simulations (only 8 simulations depicted).
Figure: Deterministic vs. Stochastic: $\nu_{\beta_I} = 1$, $\mu_{\beta_I} = \beta_I$ and $\sigma_{\beta_I} = 0.1$, with $n = 2^{15}$ Monte Carlo simulations (only 8 simulations depicted).
Numerical and statistical analysis

Technicalities

- Computer system: CPU Intel Core i7-4720HQ 2.6GHz, 16GB RAM memory and GPU GeForce GTX 970M.
- The numerical codes have been implemented in Python.
- We consider a equally spaced time grid, i.e. $\Delta t := t_{i+1} - t_i$, $\forall i$, with time step $\Delta t = \frac{1}{6}$ (around 4 hours).
- We numerically solve $n$ ODE systems, one for each path of $\tilde{\beta}_I$.
- We employ the explicit Runge-Kutta method of order 5(4), known as RK45, RKDP.
- Outcomes: the mean, the interquartile interval, $[Q_1, Q_3]$ and the worst case scenario (WS) at 95% confidence level.
Numerical and statistical analysis

Coefficients and parameters

- Initial state values given by: \( S(t_0) = N - 1, \ E(t_0) = 1 \) and

\[ I(t_0) = I_u(t_0) = H_R(t_0) = H_D(t_0) = R_d(t_0) = R_u(t_0) = D(t_0) = 0. \]

<table>
<thead>
<tr>
<th>Notation</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_I )</td>
<td>0.2887</td>
<td>Disease contact rate of a person in compartment ( I ).</td>
</tr>
<tr>
<td>( C_E )</td>
<td>0.3643</td>
<td>Reduction factor of the disease contact rate ( \beta_E ) w.r.t ( \beta_I ).</td>
</tr>
<tr>
<td>( C_u )</td>
<td>0.4010</td>
<td>Reduction factor of the disease contact rate ( \beta_I ) w.r.t ( \beta_I ).</td>
</tr>
<tr>
<td>( \delta_R )</td>
<td>7.0000</td>
<td>Difference between days in compartment ( H_R ) and ( H_D ).</td>
</tr>
<tr>
<td>( \delta_\omega )</td>
<td>0.0206</td>
<td>Difference between ( \omega ) and ( \overline{\omega} ).</td>
</tr>
<tr>
<td>( \omega )</td>
<td>0.0157</td>
<td>Lower bound of the fatality rate.</td>
</tr>
<tr>
<td>( \kappa_1 )</td>
<td>0.1082</td>
<td>Efficiency of the control measures.</td>
</tr>
</tbody>
</table>

**Table:** Parameters obtained by calibration to the data.
Numerical and statistical analysis

Coefficients and parameters

<table>
<thead>
<tr>
<th>Notation</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>1400812636</td>
<td>Total population.</td>
</tr>
<tr>
<td>$t_0$</td>
<td>1-12-2019</td>
<td>Initial date.</td>
</tr>
<tr>
<td>$T$</td>
<td>29-3-2020</td>
<td>Final date.</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>23-1-2020</td>
<td>Date when travel restrictions were imposed in Wuhan.</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>8-2-2020</td>
<td>Inflexion date.</td>
</tr>
<tr>
<td>$\theta$</td>
<td>14%</td>
<td>Percentage of documented cases at $\lambda_1$.</td>
</tr>
<tr>
<td>$\overline{\theta}$</td>
<td>65%</td>
<td>Percentage of documented cases at $\lambda_2$.</td>
</tr>
<tr>
<td>$\alpha_H$</td>
<td>2.75%</td>
<td>Percentage of infection produced by hospitalized people.</td>
</tr>
<tr>
<td>$d_E$</td>
<td>5.5</td>
<td>Average days in compartment $E$.</td>
</tr>
<tr>
<td>$d_I$</td>
<td>6.7</td>
<td>Average days in compartment $I$.</td>
</tr>
<tr>
<td>$d_{Iu}$</td>
<td>$14 - d_I = 7.3$</td>
<td>Average days in compartment $I_u$.</td>
</tr>
<tr>
<td>$d_g$</td>
<td>6</td>
<td>Maximum reduction of $d_I$ due to the control measures.</td>
</tr>
<tr>
<td>$C_0$</td>
<td>14</td>
<td>The period of convalescence.</td>
</tr>
<tr>
<td>$p(t)$</td>
<td>1</td>
<td>Fraction of the infected people hospitalized.</td>
</tr>
</tbody>
</table>

Table: Parameters extracted from the experience and/or literature.
Numerical and statistical analysis

Model variables

### 8th February, 2020 \((t = 69)\)

<table>
<thead>
<tr>
<th>Model Variables</th>
<th>(\sigma_{\beta_I} = 0)</th>
<th>(\sigma_{\beta_I} = 0.1)</th>
<th>(\sigma_{\beta_I} = 0.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E(t))</td>
<td>2993</td>
<td>3067</td>
<td>4510</td>
</tr>
<tr>
<td>(I(t))</td>
<td>1340</td>
<td>1376</td>
<td>2017</td>
</tr>
<tr>
<td>(I_u(t))</td>
<td>3854</td>
<td>3945</td>
<td>5724</td>
</tr>
<tr>
<td>(H_R(t))</td>
<td>3252</td>
<td>3328</td>
<td>4854</td>
</tr>
<tr>
<td>(H_D(t))</td>
<td>214</td>
<td>219</td>
<td>318</td>
</tr>
<tr>
<td>(R_d(t))</td>
<td>1846</td>
<td>1888</td>
<td>2726</td>
</tr>
<tr>
<td>(R_u(t))</td>
<td>4296</td>
<td>4390</td>
<td>6238</td>
</tr>
<tr>
<td>(D(t))</td>
<td>131</td>
<td>134</td>
<td>190</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Model Variables</th>
<th>(\sigma_{\beta_I} = 0)</th>
<th>(\sigma_{\beta_I} = 0.1)</th>
<th>(\sigma_{\beta_I} = 0.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E(t))</td>
<td>1</td>
<td>[1, 1]</td>
<td>2</td>
</tr>
<tr>
<td>(I(t))</td>
<td>0</td>
<td>[0, 0]</td>
<td>0</td>
</tr>
<tr>
<td>(I_u(t))</td>
<td>173</td>
<td>[146, 203]</td>
<td>259</td>
</tr>
<tr>
<td>(H_R(t))</td>
<td>232</td>
<td>[194, 272]</td>
<td>348</td>
</tr>
<tr>
<td>(H_D(t))</td>
<td>30</td>
<td>[25, 35]</td>
<td>45</td>
</tr>
<tr>
<td>(R_d(t))</td>
<td>8460</td>
<td>[7118, 9910]</td>
<td>12624</td>
</tr>
<tr>
<td>(R_u(t))</td>
<td>9969</td>
<td>[8442, 11616]</td>
<td>14681</td>
</tr>
<tr>
<td>(D(t))</td>
<td>417</td>
<td>[353, 486]</td>
<td>614</td>
</tr>
</tbody>
</table>

### 29th March, 2020 \((t = 119)\)

<table>
<thead>
<tr>
<th>Model Variables</th>
<th>(\sigma_{\beta_I} = 0)</th>
<th>(\sigma_{\beta_I} = 0.1)</th>
<th>(\sigma_{\beta_I} = 0.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E(t))</td>
<td>1</td>
<td>[1, 1]</td>
<td>2</td>
</tr>
<tr>
<td>(I(t))</td>
<td>0</td>
<td>[0, 0]</td>
<td>0</td>
</tr>
<tr>
<td>(I_u(t))</td>
<td>173</td>
<td>[146, 203]</td>
<td>259</td>
</tr>
<tr>
<td>(H_R(t))</td>
<td>232</td>
<td>[194, 272]</td>
<td>348</td>
</tr>
<tr>
<td>(H_D(t))</td>
<td>30</td>
<td>[25, 35]</td>
<td>45</td>
</tr>
<tr>
<td>(R_d(t))</td>
<td>8460</td>
<td>[7118, 9910]</td>
<td>12624</td>
</tr>
<tr>
<td>(R_u(t))</td>
<td>9969</td>
<td>[8442, 11616]</td>
<td>14681</td>
</tr>
<tr>
<td>(D(t))</td>
<td>417</td>
<td>[353, 486]</td>
<td>614</td>
</tr>
</tbody>
</table>

Table: Stochastic \(\theta\)-SEIHRD model: \(\nu_{\beta_I} = 1, \mu_{\beta_I} = \beta_I\) and \(n = 2^{15}\) Monte Carlo simulations. Columns: mean, interquartile interval \(([Q_1, Q_3])\) and worst case scenario (WS).
Numerical and statistical analysis

Model variables

\[ \sigma_{\beta I} = 0.1 \]

\[ \sigma_{\beta I} = 0.5 \]

Figure: Histogram of \( I(t) \). Setting: \( \nu_{\beta I} = 1 \) and \( \mu_{\beta I} = \beta I \), with \( n = 2^{15} \) Monte Carlo simulations.
Numerical and statistical analysis

Model outputs

- The cumulative number of COVID-19 cases, $c_m(t)$, at time $t$:

$$c_m(t) = H_R(t) + H_D(t) + R_d(t) + D(t) = c_m(t_0) + \int_{t_0}^t \theta(s)\gamma_I(s)I(s)\,ds.$$  

- The cumulative number of deaths, at time $t$: $d_m(t) = D(t)$.

- The basic reproduction number, $R_0$, and the effective reproduction number, $R_e(t)$, at time $t$, where $R_0 = R_e(t_0)$,

$$R_e(t) = \frac{U_e(t)}{\gamma_E \gamma_I(t) \gamma_HR(t) \gamma_HD(t) \gamma_Iu(t) / N},$$

with

$$U_e(t) = (**(m_Iu, H_R) + m_HR, H_R, Iu, \theta - \omega) \gamma_I + m_I, H_R, Iu) \gamma_E$$

$$+ m_E, H_R, Iu, H_D + m_HD, H_D \omega \gamma_E \gamma_I H_R, Iu.$$ 

- Hospitalized people, $H_{os}(t)$, at time $t$:

$$H_{os}(t) = H_D(t) + p(t)(H_R(t) + R_d(t) - R_d(t - C_o)).$$
Numerical and statistical analysis

Model outputs

- **Maximum number of hospitalized people in the interval** $[t_0, t]$:

  $$\text{MHos}(t) = \max_{\tau \in [t_0, t]} \text{Hos}(\tau).$$

- **The number of individuals infected by others belonging to compartments** $E$, $I_u$ and $H = H_R + H_D$:

  $$\Gamma_E(t) = \int_{t_0}^{t} m_E(s) \beta_E E(s) \frac{S(s)}{N} ds,$$

  $$\Gamma_{I_u}(t) = \int_{t_0}^{t} m_{I_u}(s) \beta_{I_u} I_u(s) \frac{S(s)}{N} ds,$$

  $$\Gamma_H(t) = \int_{t_0}^{t} (m_{H_R}(s) \beta_{H_R} H_R(s) + m_{H_D}(s) \beta_{H_D} H_D(s)) \frac{S(s)}{N} ds,$$

  respectively.
## Numerical and statistical analysis

### Model outputs

<table>
<thead>
<tr>
<th></th>
<th>8th February, 2020 (t = 69)</th>
<th>29th March, 2020 (t = 119)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_{\beta_I} = 0$</td>
<td>$\sigma_{\beta_I} = 0.1$</td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>[Q&lt;sub&gt;1&lt;/sub&gt;, Q&lt;sub&gt;3&lt;/sub&gt;]</td>
</tr>
<tr>
<td>$c_m(t)$</td>
<td>5440</td>
<td>[4586, 6362]</td>
</tr>
<tr>
<td>$d_m(t)$</td>
<td>131</td>
<td>[112, 152]</td>
</tr>
<tr>
<td>$R_e(t)$</td>
<td>0.33</td>
<td>[0.31, 0.35]</td>
</tr>
<tr>
<td>$Hos(t)$</td>
<td>4040</td>
<td>[3395, 4727]</td>
</tr>
<tr>
<td>$MHos(t)$</td>
<td>4040</td>
<td>[3395, 4727]</td>
</tr>
<tr>
<td>$\Gamma_E(t)$</td>
<td>5012</td>
<td>[4255, 5833]</td>
</tr>
<tr>
<td>$\Gamma_{IU}(t)$</td>
<td>4550</td>
<td>[3864, 5285]</td>
</tr>
<tr>
<td>$\Gamma_H(t)$</td>
<td>198</td>
<td>[168, 230]</td>
</tr>
</tbody>
</table>

### Table: Stochastic $\theta$-SEIHRD model

$\nu_{\beta_I} = 1$, $\mu_{\beta_I} = \beta_I$ and $n = 2^{15}$ Monte Carlo simulations. Columns: mean, interquartile interval ([Q<sub>1</sub>, Q<sub>3</sub>]) and worst case scenario (WS).

Alvaro Leitao and Carlos Vázquez
Numerical and statistical analysis
Evolution epidemic curves

Figure: Epidemic curves: mean, IQ interval and WS. Setting: $\nu_{\beta_l} = 1$, $\mu_{\beta_l} = \beta_l$ and $\sigma_{\beta_l} = 0.1$, with $n = 2^{15}$. 

Álvaro Leitao and Carlos Vázquez
Numerical and statistical analysis
Evolution epidemic curves

Figure: Epidemic curves: mean, IQ interval and WS. Setting: \( \nu \beta_I = 1, \mu \beta_I = \beta_I \) and \( \sigma \beta_I = 0.5 \), with \( n = 2^{15} \).
We have extended an ad-hoc model by incorporating randomness to relevant coefficients (disease contact rates).

We have shown the importance of considering the uncertainty.

The presented modelling approach is more complete since it allows to compute confidence intervals and worst case scenarios.

The information provided by the worst case scenarios can be useful to develop more conservative policies in the actions against the COVID-19 spread.

A natural extension would be to consider independent contact rates, using different stochastic processes to characterize their dynamics.

In this more general setting, a certain number of different (possibly correlated) Brownian motion processes would come into place.
A theory of the term structure of interest rates.  

Benjamin Ivorra, Miriam R. Ferrández, María Vela-Pérez, and Ángel M. Ramos.  

Álvaro Leitao and Carlos Vázquez.  
Acknowledgements

This work was funded by Xunta de Galicia grant ED431C2018/033, including FEDER funding. Both authors also acknowledge the support received from the Centro de Investigación de Galicia “CITIC”, funded by Xunta de Galicia and the European Union (European Regional Development Fund- Galicia 2014-2020 Program), by grant ED431G 2019/01. Álvaro Leitao acknowledges the financial support from the Spanish Ministry of Science, Innovation and Universities, through the Juan de la Cierva-formación 2017 (FJC17) grant in the framework of the national programme for R&D 2013-2016.

Thank you for your attention

More: alvaroleitao.github.io