Efficient one and multiple time-step Monte Carlo simulation of the SABR model

Álvaro Leitao, Lech A. Grzelak and Cornelis W. Oosterlee

Delft University of Technology - Centrum Wiskunde & Informatica

Frankfurt - April 20, 2017

"Our" definition of simulation

- Generate samples from (sampling) stochastic processes.
- The standard approach to sample from a given distribution, Z:

$$F_Z(Z) \stackrel{\mathrm{d}}{=} U \quad \text{thus} \quad z_n = F_Z^{-1}(u_n),$$

- F_Z is the cumulative distribution function (CDF).
- $\stackrel{\mathrm{d}}{=}$ means equality in the distribution sense.
- $U \sim \mathcal{U}([0,1])$ and u_n is a sample from $\mathcal{U}([0,1])$.
- The computational cost depends on inversion F_Z^{-1}.

Outline

1. SABR model
2. Distribution of the SABR's integrated variance
3. One-step SABR simulation
4. Multiple time-step SABR simulation
5. Conclusions

SABR model

- The formal definition of the SABR model [6] reads

$$\begin{aligned} \mathrm{d}S(t) &= \sigma(t)S^{\beta}(t)\mathrm{d}W_S(t), & S(0) &= S_0 \exp\left(rT\right), \\ \mathrm{d}\sigma(t) &= \alpha\sigma(t)\mathrm{d}W_{\sigma}(t), & \sigma(0) &= \sigma_0. \end{aligned}$$

- $S(t) = \bar{S}(t)\exp\left(r(T-t)\right)$ is the forward price of the underlying $\bar{S}(t)$, with r an interest rate, S_0 the spot price and T the maturity.
- $\sigma(t)$ is the stochastic volatility.
- $W_f(t)$ and $W_{\sigma}(t)$ are two correlated Brownian motions.
- SABR parameters:
  - The volatility of the volatility, $\alpha > 0$.
  - The CEV elasticity, $0 \leq \beta \leq 1$.
  - The correlation coefficient, ρ ($W_f W_{\sigma} = \rho t$).

"Exact" simulation of SABR model

- Based on Islah [7], the conditional cumulative distribution function (CDF) of $S(t)$ in a generic interval $[s,t]$, $0 \leq s \leq t \leq T$:

$$Pr\left(S(t) \leq K | S(s) > 0, \sigma(s), \sigma(t), \int_s^t \sigma^2(z)\mathrm{d}z\right) = 1 - \chi^2(a; b, c),$$

where

$$a = \frac{1}{\nu(t)}\left(\frac{S(s)^{1-\beta}}{(1-\beta)} + \frac{\rho}{\alpha}\left(\sigma(t) - \sigma(s)\right)\right)^2,$$

$$c = \frac{K^{2(1-\beta)}}{(1-\beta)^2\nu(t)},$$

$$b = 2 - \frac{1 - 2\beta - \rho^2(1-\beta)}{(1-\beta)(1-\rho^2)},$$

$$\nu(t) = (1-\rho^2)\int_s^t \sigma^2(z)\mathrm{d}z,$$

and $\chi^2(x; \delta, \lambda)$ is the non-central chi-square CDF.

- Exact in the case of $\rho = 0$, an *approximation* otherwise.

Simulation of SABR model

- Simulation of the volatility process, $\sigma(t)|\sigma(s)$:

$$\sigma(t) \sim \sigma(s) \exp(\alpha \hat{W}_\sigma(t) - \frac{1}{2}\alpha^2 t),$$

  where $\hat{W}_\sigma(t)$ is a independent Brownian motion.
- Simulation of the integrated variance process, $\int_s^t \sigma^2(z)\mathrm{d}z|\sigma(t), \sigma(s)$.
- Simulation of the forward process, $S(t)|S(s), \int_s^t \sigma^2(z)\mathrm{d}z, \sigma(t), \sigma(s)$ by inverting the CDF.
- The conditional integrated variance is a challenging part. We propose:
  - Approximate the conditional distribution by using Fourier techniques and copulas.
  - Marginal distribution based on COS method [4].
  - Conditional distribution based on copulas.
  - Improvements in performance and efficiency.

Distribution of the integrated variance

- Not available.
- For notational convenience, we will use $Y(s,t) := \int_s^t \sigma^2(z)\mathrm{d}z$.
- Discrete equivalent, M monitoring dates:

$$Y(s,t) := \int_s^t \sigma^2(z)\mathrm{d}z \approx \sum_{j=1}^{M} \Delta t \sigma^2(t_j) =: \hat{Y}(s,t)$$

  where $t_j = s + j\Delta t$, $j = 1, \ldots, M$ and $\Delta t = \frac{t-s}{M}$.
- In the logarithmic domain, where we aim to find an approximation of $F_{\log \hat{Y} | \log \sigma(s)}$:

$$F_{\log \hat{Y} | \log \sigma(s)}(x) = \int_{-\infty}^{x} f_{\log \hat{Y} | \log \sigma(s)}(y)\mathrm{d}y,$$

  where $f_{\log \hat{Y} | \log \sigma(s)}$ is the *probability density function* (PDF) of $\log \hat{Y}(s,t) | \log \sigma(s)$.

PDF of the integrated variance

- Equivalent: Characteristic function and inversion (Fourier pair).
- Recursive procedure to derive an approximated $\phi_{\log \hat{Y} | \log \sigma(s)}$.
- We start by defining the logarithmic increment of $\sigma^2(t)$:

$$R_j = \log\left(\frac{\sigma^2(t_j)}{\sigma^2(t_{j-1})}\right), j = 1, \ldots, M.$$

- $\sigma^2(t_j)$ can be written:

$$\sigma^2(t_j) = \sigma^2(t_0)\exp(R_1 + R_2 + \cdots + R_j).$$

- We introduce the iterative process

$$\begin{aligned} Y_1 &= R_M, \\ Y_j &= R_{M+1-j} + Z_{j-1}, \quad j = 2, \ldots, M. \end{aligned}$$

with $Z_j = \log(1 + \exp(Y_j))$.

PDF of the integrated variance (cont.)

- $\hat{Y}(s,t)$ can be expressed:

$$\hat{Y}(s,t) = \sum_{i=1}^{M} \sigma^2(t_i)\Delta t = \Delta t \sigma^2(s) \exp(Y_M).$$

- And, we compute $\phi_{\log \hat{Y} | \log \sigma(s)}(u)$, as follows:

$$\phi_{\log \hat{Y} | \log \sigma(s)}(u) = \exp\left(iu \log(\Delta t \sigma^2(s))\right) \phi_{Y_M}(u).$$

- By applying COS method [4] in the support $[\hat{a}, \hat{b}]$:

$$f_{\log \hat{Y} | \log \sigma(s)}(x) \approx \frac{2}{\hat{b}-\hat{a}} \sum_{k=0}^{N-1'} C_k \cos\left((x-\hat{a})\frac{k\pi}{\hat{b}-\hat{a}}\right),$$

with

$$C_k = \Re\left(\phi_{\log \hat{Y} | \log \sigma(s)}\left(\frac{k\pi}{\hat{b}-\hat{a}}\right) \exp\left(-i\frac{\hat{a}k\pi}{\hat{b}-\hat{a}}\right)\right).$$

CDF of the integrated variance

- The CDF of $\log \hat{Y}(s,t) | \log \sigma(s)$:

$$F_{\log \hat{Y} | \log \sigma(s)}(x) = \int_{-\infty}^{x} f_{\log \hat{Y} | \log \sigma(s)}(y) \mathrm{d}y$$
$$\approx \int_{\hat{a}}^{x} \frac{2}{\hat{b} - \hat{a}} \sum_{k=0}^{N-1'} C_k \cos\left((y - \hat{a}) \frac{k\pi}{\hat{b} - \hat{a}} \right) \mathrm{d}y.$$

- The efficient computation of $\phi_{\log \hat{Y} | \log \sigma(s)}$ is crucial for the performance of the whole procedure (specially, one-step case).
- The inversion of $F_{\log \hat{Y} | \log \sigma(s)}$ is relatively expensive (unafforable in the multi-step case).

Copula-based simulation of $\int_s^t \sigma^2(z)\mathrm{d}z | \sigma(t), \sigma(s)$

- In order to apply copulas, we need (logarithmic domain):
  - $F_{\log \hat{Y} | \log \sigma(s)}$.
  - $F_{\log \sigma(t) | \log \sigma(s)}$.
  - Correlation between $\log Y(s,t)$ and $\log \sigma(t)$.
- The distribution of $\log \sigma(t) | \log \sigma(s)$ is known ($\sigma(t)$ follows a log-normal distribution).
- Approximated Pearson's correlation coefficient:

$$\mathcal{P}_{\log Y, \log \sigma(t)} \approx \frac{t^2 - s^2}{2\sqrt{\left(\frac{1}{3}t^4 + \frac{2}{3}ts^3 - t^2 s^2\right)}}.$$

- For some copulas, like Archimedean, Kendall's τ is required:

$$\mathcal{P} = \sin\left(\frac{\pi}{2}\tau\right).$$

Sampling $\int_s^t \sigma^2(z)\mathrm{d}z|\sigma(t),\sigma(s)$: Steps

1. Determine $F_{\log\sigma(t)|\log\sigma(s)}$ and $F_{\log\hat{Y}|\log\sigma(s)}$.
2. Determine the correlation between $\log Y(s,t)$ and $\log\sigma(t)$.
3. Generate correlated uniform samples, $U_{\log\sigma(t)|\log\sigma(s)}$ and $U_{\log\hat{Y}|\log\sigma(s)}$ by means of copula.
4. From $U_{\log\sigma(t)|\log\sigma(s)}$ and $U_{\log\hat{Y}|\log\sigma(s)}$ invert original marginal distributions.
5. The samples of $\sigma(t)|\sigma(s)$ and $Y(s,t)=\int_s^t \sigma^2(z)\mathrm{d}z|\sigma(t),\sigma(s)$ are obtained by taking exponentials.

One time-step simulation of the SABR model

- $s = 0$ and $t = T$, with T the maturity time.
- The use is restricted to price European options up to $T = 2$.
- $\log \sigma(s)$ becomes constant.
- $F_{\log \sigma(t) | \log \sigma(s)}$ and $F_{\log \hat{Y} | \log \sigma(s)}$ turn into $F_{\log \sigma(T)}$ and $F_{\log \hat{Y}(T)}$.
- The computation of $\phi_{\log \hat{Y}(T)}$ is much simpler and very fast.
- The approximated Pearson's coefficient results in a constant value:

$$\mathcal{P}_{\log Y(T), \log \sigma(T)} \approx \frac{T^2}{2\sqrt{\frac{1}{3}T^4}} = \frac{\sqrt{3}}{2}.$$

Approximated correlation

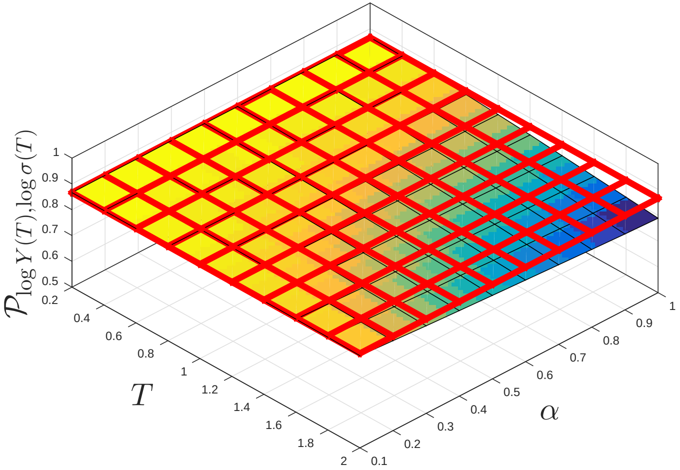


Figure: Pearson's coefficient: Empirical (surface) vs. approximation (red grid).

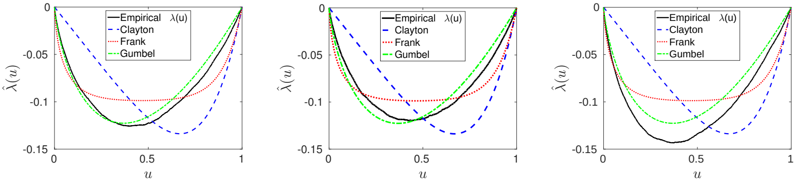
Copula analysis

- Based on the one-step simulation, a copula analysis is carried out.
- Gaussian, Student **t** and Archimedean (Clayton, Frank and Gumbel).
- A *goodness-of-fit (GOF)* for copulas needs to be evaluated.
- Archimedean: graphic GOF based on Kendall's processes.
- Generic GOF based on the so-called *Deheuvels or empirical copula*.

	S_0	σ_0	α	β	ρ	T
Set I	1.0	0.5	0.4	0.7	0.0	2
Set II	0.05	0.1	0.4	0.0	-0.8	0.5
Set III	0.04	0.4	0.8	1.0	-0.5	2

Table: Data sets.

GOF - Archimedean



(a) Set I. (b) Set II. (c) Set III.

Figure: Archimedean GOF test: $\hat{\lambda}(u)$ vs. empirical $\lambda(u)$.

	Clayton	Frank	Gumbel
Set I	1.3469×10^{-3}	2.9909×10^{-4}	5.1723×10^{-5}
Set II	1.0885×10^{-3}	2.1249×10^{-4}	8.4834×10^{-5}
Set III	2.1151×10^{-3}	7.5271×10^{-4}	2.6664×10^{-4}

Table: MSE of $\hat{\lambda}(u) - \lambda(u)$.

Generic GOF

	Gaussian	Student $\mathbf{t}$ $(\nu = 5)$	Gumbel
Set I	5.0323×10^{-3}	5.0242×10^{-3}	3.8063×10^{-3}
Set II	3.1049×10^{-3}	3.0659×10^{-3}	4.5703×10^{-3}
Set III	5.9439×10^{-3}	6.0041×10^{-3}	4.3210×10^{-3}

Table: Generic GOF: D_2.

- The three copulas perform very similarly.
- For longer maturities: Gumbel performs better.
- The Student $\mathbf{t}$ copula is discarded: very similar to the Gaussian copula and the calibration of the ν parameter adds extra complexity.
- As a general strategy, the Gumbel copula is the most robust choice.
- With short maturities, the Gaussian copula may be a satisfactory alternative.

Pricing European options

- The strike values K_i are chosen following the expression:

$$K_i(T) = S(0)\exp(0.1 \times T \times \delta_i),$$
$$\delta_i = -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5.$$

- Forward asset, $S(t)$: enhanced inversion by Chen et al. [3].
- Martingale correction:

$$\begin{aligned} S(t) &= S(t) - \frac{1}{n}\sum_{i=1}^{n} S_i(t) + \mathbb{E}[S(t)], \\ &= S(t) - \frac{1}{n}\sum_{i=1}^{n} S_i(t) + S_0, \end{aligned}$$

Pricing European options - Convergence and time

	$n = 1000$	$n = 10000$	$n = 100000$	$n = 1000000$
	Gaussian (Set I, K_1)			
Error	519.58	132.39	37.42	16.23
Time	0.3386	0.3440	0.3857	0.5733
	Gumbel (Set I, K_1)			
Error	151.44	-123.76	34.14	11.59
Time	0.3492	0.3561	0.3874	0.6663

Table: Convergence in number of samples, n: error (basis points) and execution time (*sec.*).

Pricing European options - Implied volatilities

Strikes	K_1	K_2	K_3	K_4	K_5	K_6	K_7
	Set I (Reference: Antonov [1])						
Hagan	55.07	52.34	50.08	N/A	47.04	46.26	45.97
MC	23.50	21.41	19.38	N/A	16.59	15.58	14.63
Gaussian	16.23	20.79	24.95	N/A	33.40	37.03	40.72
Gumbel	11.59	15.57	19.12	N/A	25.41	28.66	31.79
	Set II (Reference: Korn [8])						
Hagan	−558.82	−492.37	−432.11	−377.47	−327.92	−282.98	−242.22
MC	5.30	6.50	7.85	9.32	10.82	12.25	13.66
Gaussian	9.93	9.98	10.02	10.20	10.57	10.73	11.04
Gumbel	−9.93	−9.38	−8.94	−8.35	−7.69	−6.83	−5.79
	Set III (Reference: MC Milstein)						
Hagan	287.05	252.91	220.39	190.36	163.87	141.88	126.39
Gaussian	16.10	16.76	16.62	15.22	13.85	12.29	10.67
Gumbel	6.99	3.79	0.67	−2.27	−5.57	−9.79	−14.06

Table: Implied volatility: errors in basis points.

- One-step SABR simulation is a fast alternative to Hagan formula.
- Overcomes the known issues, like low strikes and high volatilities.
- For longer maturities and more complex options: multiple time-step

Multiple time-step simulation of the SABR model

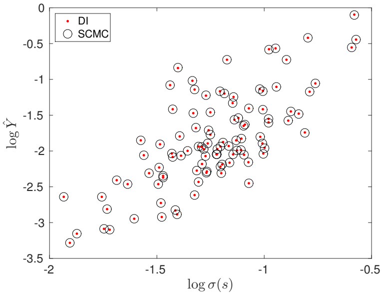
- We denote it *mSABR* simulation method (scheme).
- In intermediate steps, $\phi_{\log \hat{Y}|\log\sigma(s)}$ becomes "stochastic".
- $f_{\log \hat{Y}|\log\sigma(s)}$ needs to be computed for each sample of $\log\sigma(s)$.
- Consequently, the inversion of $F_{\log \hat{Y}|\log\sigma(s)}$ is unaffordable ($n \uparrow\uparrow$).
- Solution: *Stochastic Collocation Monte Carlo* (SCMC) sampler [5].

$$y_n|v_n \approx g_{L_{\hat{Y}},L_\sigma}(x_n) = \sum_{i=1}^{L_{\hat{Y}}} \sum_{j=1}^{L_\sigma} F^{-1}_{\log \hat{Y}|\log\sigma(s)=v_j}(F_X(x_i))\ell_i(x_n)\ell_j(v_n),$$

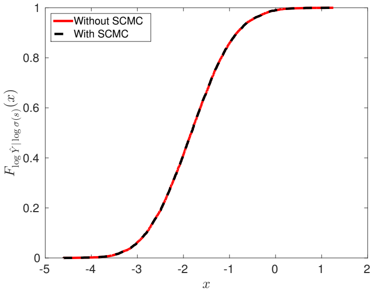
where x_n are the samples from the *cheap variable*, X, and v_n the given samples of $\log\sigma(s)$. x_i and v_j are the *collocation points* of X and $\log\sigma(s)$, respectively. ℓ_i and ℓ_j are the Lagrange polynomials defined by

$$\ell_i(x_n) = \prod_{k=1,k\neq i}^{L_{\hat{Y}}} \frac{x_n - x_k}{x_i - x_k}, \quad \ell_j(v_n) = \prod_{k=1,k\neq j}^{L_\sigma} \frac{v_n - v_k}{v_i - v_k}.$$

Application of 2D SCMC to $F_{\log \hat{Y} | \log \sigma(s)}$



(a) $\log \hat{Y} | \log \sigma(s)$ - DI. vs. SCMC.



(b) $F_{\log \hat{Y} | \log \sigma(s)}(x)$.

Samples	Without SCMC	With SCMC		
		$L_{\hat{Y}} = L_\sigma = 3$	$L_{\hat{Y}} = L_\sigma = 7$	$L_{\hat{Y}} = L_\sigma = 11$
100	1.0695	0.0449	0.0466	0.0660
10000	16.3483	0.0518	0.0588	0.0798
1000000	1624.3019	0.2648	0.5882	1.0940

Table: SCMC time

mSABR method - Experiments

- The strike values K_i are chosen following the expression:

$$K_i(T) = S(0)\exp(0.1 \times T \times \delta_i),$$
$$\delta_i = -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5.$$

- Forward asset, $S(t)$: enhanced inversion by Chen et al. [3].
- Martingale correction:

$$S(t) = S(t) - \frac{1}{n}\sum_{i=1}^{n} S_i(t) + S_0,$$

- New data sets:

	S_0	σ_0	α	β	ρ	T
Set I [5]	0.5	0.5	0.4	0.5	0.0	4
Set II [3]	0.04	0.2	0.3	1.0	−0.5	5
Set III [1]	1.0	0.25	0.3	0.6	−0.5	20
Set IV [2]	0.0056	0.011	1.080	0.167	0.999	1

Table: Data sets.

mSABR method - Convergence test I

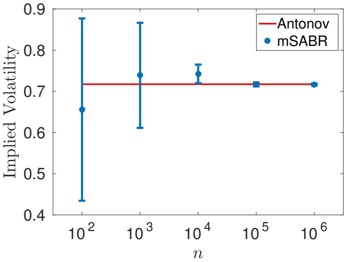
- Convergence in number of time-steps, m: Antonov vs. mSABR. Set I.

Strikes	K_1	K_2	K_3	K_4	K_5	K_6	K_7
Antonov	73.34%	71.73%	70.17%	N/A	67.23%	65.87%	64.59%
$m = T/4$	73.13%	71.75%	70.41%	69.11%	67.85%	66.64%	65.48%
Error(bp)	-21.51	2.54	24.38	N/A	61.71	76.66	89.26
$m = T/2$	73.30%	71.78%	70.29%	68.86%	67.49%	66.17%	64.93%
Error(bp)	-4.12	4.94	12.71	N/A	25.48	30.40	34.73
$m = T$	73.25%	71.67%	70.14%	68.66%	67.24%	65.89%	64.62%
Error(bp)	-9.56	-5.93	-2.79	N/A	0.92	2.21	3.17
$m = 2T$	73.32%	71.71%	70.16%	68.65%	67.22%	65.85%	64.55%
Error(bp)	-2.08	-1.56	-1.20	N/A	-1.65	-2.35	-3.36
$m = 4T$	73.34%	71.73%	70.18%	68.67%	67.24%	65.87%	64.58%
Error(bp)	0.15	0.58	0.78	N/A	0.43	0.04	-0.48

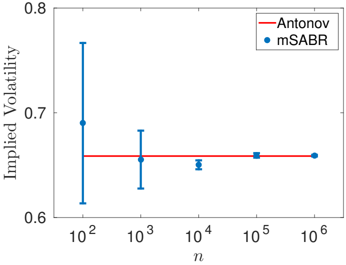
mSABR method - Convergence test II

- Convergence in number of samples, n: Antonov vs. mSABR. Set I.

Strikes	K_1	K_2	K_3	K_4	K_5	K_6	K_7
Antonov	73.34%	71.73%	70.17%	N/A	67.23%	65.87%	64.59%
$n = 10^2$	67.29%	65.55%	63.84%	62.20%	60.63%	59.01%	57.65%
RE	8.24×10^{-2}	8.61×10^{-2}	9.01×10^{-2}	N/A	9.82×10^{-2}	1.04×10^{-1}	1.07×10^{-1}
$n = 10^4$	73.41%	71.87%	70.36%	68.91%	67.51%	66.19%	64.94%
RE	9.65×10^{-4}	1.94×10^{-3}	2.75×10^{-3}	N/A	4.08×10^{-3}	4.93×10^{-3}	5.48×10^{-3}
$n = 10^6$	73.34%	71.73%	70.18%	68.67%	67.24%	65.87%	64.58%
RE	2.04×10^{-5}	8.08×10^{-5}	1.11×10^{-4}	N/A	6.39×10^{-5}	6.07×10^{-6}	7.43×10^{-5}



(c) Strike K_2.



(d) Strike K_6.

mSABR method - Stability in ρ

- Implied volatility, varying ρ: Monte Carlo (MC) vs. mSABR. Set II.

Strikes	K_1	K_2	K_3	K_4	K_5	K_6	K_7
				$\rho = -0.5$			
MC	22.17%	21.25%	20.38%	19.57%	18.88%	18.33%	17.95%
mSABR	22.21%	21.28%	20.39%	19.58%	18.88%	18.32%	17.94%
Error(bp)	3.59	2.86	1.78	0.95	-0.19	-0.96	-1.10
				$\rho = 0.0$			
MC	21.35%	20.96%	20.71%	20.63%	20.71%	20.96%	21.34%
mSABR	21.35%	20.95%	20.69%	20.60%	20.68%	20.93%	21.32%
Error(bp)	0.04	-1.04	-2.51	-3.02	-3.33	-3.19	-2.56
				$\rho = 0.5$			
MC	19.66%	20.04%	20.61%	21.34%	22.20%	23.14%	24.16%
mSABR	19.59%	19.96%	20.54%	21.28%	22.15%	23.11%	24.11%
Error(bp)	-6.93	-7.36	-6.77	-5.53	-4.35	-3.76	-4.05

mSABR method - Performance

- But, is it worth to use the mSABR method?

Error	< 100 bp	< 50 bp	< 25 bp	< 10 bp
MC Euler	6.85(200)	10.71(300)	27.42(800)	42.90(1200)
Y-Euler	2.18(4)	6.55(16)	11.85(32)	45.12(128)
Y-trpz	2.17(3)	4.24(8)	7.25(16)	14.47(32)
mSABR	3.46(1)	2.98(2)	3.72(3)	4.89(4)

Table: Execution times and time-steps, m (parentheses).

Error	< 100 bp	< 50 bp	< 25 bp	< 10 bp
MC Euler	1.98	3.59	7.37	8.77
Y-Euler	0.63	2.19	3.18	9.22
Y-trpz	0.62	1.42	1.94	2.95

Table: Speedups provided by the mSABR method.

mSABR method - Pricing barrier options

- The *up-and-out* call option is considered here
- The price, with the barrier level, B, $B > S_0, B > K_i$, reads:

$$V_i(K_i, B, T) = \exp(-rT)\,\mathbb{E}\left[(S(T) - K_i)\mathbb{1}(\max_{0<t_k\leq T} S(t_k) > B)\right],$$

where t_k are the times where the barrier condition is checked.
- Setting: $n = 10^6$ and $m = 4T$.
- We define the *mean squared error* (MSE) as

$$\text{MSE} = \frac{1}{7}\sum_{i=1}^{7}\left(V_i^{MC}(K_i, B, T) - V_i^{mSABR}(K_i, B, T)\right)^2$$

where $V_i^{MC}(K_i, B, T)$ and $V_i^{mSABR}(K_i, B, T)$ are the barrier option prices provided by standard Monte Carlo method and by the mSABR method, respectively.

mSABR method - Pricing barrier options

- Pricing barrier options with mSABR: $V_i(K_i, B, T) \times 100$. Set II:

Strikes	K_1	K_2	K_3	K_4	K_5	K_6	K_7
				$B = 0.08$			
MC	1.1702	0.9465	0.7268	0.5215	0.3423	0.1996	0.0987
mSABR	1.1724	0.9486	0.7285	0.5226	0.3428	0.1997	0.0986
MSE	1.8910×10^{-10}						
				$B = 0.1$			
MC	1.3099	1.0766	0.8462	0.6290	0.4367	0.2794	0.1626
mSABR	1.3092	1.0761	0.8456	0.6282	0.4355	0.2782	0.1618
MSE	7.5542×10^{-11}						
				$B = 0.12$			
MC	1.3521	1.1168	0.8841	0.6644	0.4695	0.3093	0.1891
mSABR	1.3518	1.1166	0.8838	0.6639	0.4686	0.3080	0.1880
MSE	6.3648×10^{-11}						

- Pricing barrier options with mSABR: $V_i(K_i, B, T) \times 100$. Set III:

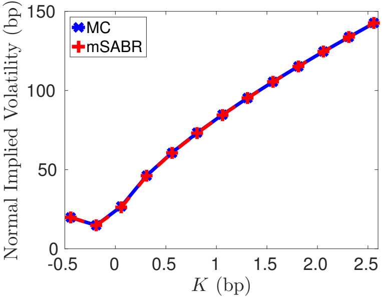
Strikes	K_1	K_2	K_3	K_4	K_5	K_6	K_7
				$B = 2.0$			
MC	29.1174	23.4804	17.2273	10.7825	5.0203	1.1750	0.0036
mSABR	29.2346	23.5828	17.3086	10.8327	5.0385	1.1805	0.0036
MSE	4.8146×10^{-7}						
				$B = 2.5$			
MC	41.3833	34.5497	26.8311	18.6089	10.7281	4.4893	0.9434
mSABR	41.3394	34.5097	26.7948	18.5747	10.6943	4.4546	0.9320
MSE	1.2131×10^{-7}						
				$B = 3.0$			
MC	48.5254	41.1652	32.7980	23.7807	14.9344	7.5364	2.6692
mSABR	48.5008	41.1515	32.7888	23.7655	14.9097	7.5117	2.6549
MSE	3.6201×10^{-8}						

mSABR method - Negative interest rates

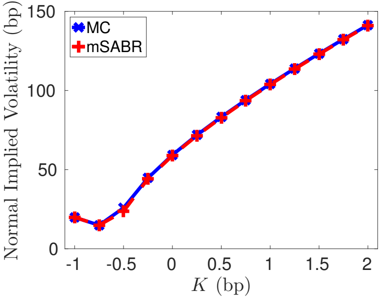
- The mSABR method in combination with the shifted SABR model:

$$\mathrm{d}S(t) = \sigma(t)(S(t)+\theta)^{\beta}\mathrm{d}W_S(t),$$
$$S(0) = (S_0+\theta)\exp(rT),$$

  where $\theta > 0$ is a displacement, or shift, in the underlying.
- Setting: $n = 10^6$, $m = 4T$ and $\theta = 0.02$.



(e) Set IV



(f) Set IV: $S_0 = 0$

Conclusions

- We propose an efficient SABR simulation based on Fourier and copula techniques.
- The one-step SABR is a fast alternative to Hagan formula for short maturities.
- Overcomes the known issues of Hagan's expression.
- When longer maturities and/or more involved options are considered, multi-step version.
- High accuracy with very few number of time-steps, even in the context of negative interest rates.
- Good balance between accuracy and computational cost.

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Acknowledgments

This research was supported by the European Union in the FP7-PEOPLE-2012-ITN Program under Grant Agreement Number 304617 (FP7 Marie Curie Action, Project Multi-ITN STRIKE – Novel Methods in Computational Finance).

Thank you for your attention