Efficient one and multiple time-step simulation of the SABR model

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“Our” definition of simulation

- Generate samples from (sampling) stochastic processes.
- The standard approach to sample from a given distribution, Z:

$$F_Z(Z) \stackrel{\mathrm{d}}{=} U \quad \text{thus} \quad z_n = F_Z^{-1}(u_n),$$

- F_Z is the cumulative distribution function (CDF).
- $\stackrel{\mathrm{d}}{=}$ means equality in the distribution sense.
- $U \sim \mathcal{U}([0,1])$ and u_n is a sample from $\mathcal{U}([0,1])$.
- The computational cost depends on inversion F_Z^{-1}.

Outline

SABR model

- The formal definition of the SABR model [5] reads

$$\begin{aligned} \mathrm{d}f(t) &= \sigma(t) f^{\beta}(t) \mathrm{d}W_f(t), \quad f(0) = S_0, \\ \mathrm{d}\sigma(t) &= \alpha\sigma(t)\mathrm{d}W_\sigma(t), \qquad \sigma(0) = \sigma_0, \end{aligned}$$

- $f(t) = S(t)e^{rt}$ is forward price of the underlying asset $S(t)$.
- $\sigma(t)$ is the stochastic volatility.
- $W_f(t)$ and $W_\sigma(t)$ are two correlated Brownian motions
- SABR parameters:
  - The volatility of the volatility, $\alpha > 0$.
  - The CEV elasticity, $0 \leq \beta \leq 1$.
  - The correlation coefficient, ρ ($W_f W_\sigma = \rho t$)

"Exact" simulation of SABR model

- Based on Islah [6], the conditional cumulative distribution function (CDF) of $f(t)$ in a generic interval $[s,t]$, $0 \leq s \leq t \leq T$:

$$Pr\left(f(t) \leq K | f(s) > 0, \sigma(s), \sigma(t), \int_s^t \sigma^2(z)\mathrm{d}z\right) = 1 - \chi^2(a; b, c),$$

where

$$a = \frac{1}{\nu(t)}\left(\frac{f(s)^{1-\beta}}{(1-\beta)} + \frac{\rho}{\alpha}\left(\sigma(t) - \sigma(s)\right)\right)^2,$$

$$c = \frac{K^{2(1-\beta)}}{(1-\beta)^2\nu(t)},$$

$$b = 2 - \frac{1 - 2\beta - \rho^2(1-\beta)}{(1-\beta)(1-\rho^2)},$$

$$\nu(t) = (1-\rho^2)\int_s^t \sigma^2(z)\mathrm{d}z,$$

and $\chi^2(x;\delta,\lambda)$ is the non-central chi-square CDF.

- Exact in the case of $\rho = 0$, an *approximation* otherwise.

Simulation of SABR model

- Simulation of the volatility process, $\sigma(t)|\sigma(s)$:

$$\sigma(t) \sim \sigma(s)\exp(\alpha \hat{W}_\sigma(t) - \frac{1}{2}\alpha^2 t),$$

  where $\hat{W}_\sigma(t)$ is a independent Brownian motion.
- Simulation of the integrated variance process, $\int_s^t \sigma^2(z)\mathrm{d}z|\sigma(t),\sigma(s)$.
- Simulation of the forward process, $f(t)|f(s), \int_s^t \sigma^2(z)\mathrm{d}z, \sigma(t), \sigma(s)$ by inverting the CDF.
- The conditional integrated variance is a challenging part. We propose:
  - Approximate the conditional distribution by using Fourier techniques and copulas.
  - Marginal distribution based on COS method [3].
  - Conditional distribution based on copulas.
  - Improvements for a fast computations.

Distribution of the integrated variance

- Not available.
- For notational convenience, we will use $Y(s,t) := \int_s^t \sigma^2(z)\mathrm{d}z$.
- Discrete equivalent, M monitoring dates:

$$Y(s,t) := \int_s^t \sigma^2(z)\mathrm{d}z \approx \sum_{j=1}^{M} \Delta t \sigma^2(t_j) =: \hat{Y}(s,t)$$

  where $t_j = s + j\Delta t$, $j = 1, \ldots, M$ and $\Delta t = \frac{t-s}{M}$.
- In the logarithmic domain, where we aim to find an approximation of $F_{\log \hat{Y} | \log \sigma(s)}$:

$$F_{\log \hat{Y} | \log \sigma(s)}(x) = \int_{-\infty}^{x} f_{\log \hat{Y} | \log \sigma(s)}(y)\mathrm{d}y,$$

  where $f_{\log \hat{Y} | \log \sigma(s)}$ is the *probability density function* (PDF) of $\log \hat{Y}(s,t) | \log \sigma(s)$.

PDF of the integrated variance

- Equivalent: Characteristic function and inversion (Fourier pair).
- Recursive procedure to derive an approximated $\phi_{\log \hat{Y} | \log \sigma(s)}$.
- We start by defining the logarithmic increment of $\sigma^2(t)$:

$$R_j = \log\left(\frac{\sigma^2(t_j)}{\sigma^2(t_{j-1})}\right), j = 1, \ldots, M$$

- $\sigma^2(t_j)$ can be written:

$$\sigma^2(t_j) = \sigma^2(t_0)\exp(R_1 + R_2 + \cdots + R_j).$$

- We introduce the iterative process

$$\begin{aligned} Y_1 &= R_M, \\ Y_j &= R_{M+1-j} + Z_{j-1}, \quad j = 2, \ldots, M. \end{aligned}$$

with $Z_j = \log(1 + \exp(Y_j))$.

PDF of the integrated variance (cont.)

- $\hat{Y}(s,t)$ can be expressed:

$$\hat{Y}(s,t) = \sum_{i=1}^{M} \sigma^2(t_i)\Delta t = \Delta t \sigma^2(s) \exp(Y_M).$$

- And, we compute $\phi_{\log \hat{Y} | \log \sigma(s)}(u)$, as follows:

$$\phi_{\log \hat{Y} | \log \sigma(s)}(u) = \exp\left(iu \log(\Delta t \sigma^2(s))\right) \phi_{Y_M}(u).$$

- By applying COS method in the support $[\hat{a}, \hat{b}]$:

$$f_{\log \hat{Y} | \log \sigma(s)}(x) \approx \frac{2}{\hat{b}-\hat{a}} \sum_{k=0}^{N-1}{}' C_k \cos\left((x-\hat{a}) \frac{k\pi}{\hat{b}-\hat{a}} \right),$$

with

$$C_k = \Re\left(\phi_{\log \hat{Y} | \log \sigma(s)}\left(\frac{k\pi}{\hat{b}-\hat{a}}\right) \exp\left(-i \frac{\hat{a}k\pi}{\hat{b}-\hat{a}}\right)\right).$$

CDF of the integrated variance

- The CDF of $\log \hat{Y}(s,t) | \log \sigma(s)$:

$$F_{\log \hat{Y} | \log \sigma(s)}(x) = \int_{-\infty}^{x} f_{\log \hat{Y} | \log \sigma(s)}(y) \mathrm{d}y$$

$$\approx \int_{\hat{a}}^{x} \frac{2}{\hat{b} - \hat{a}} \sum_{k=0}^{N-1\,'} C_k \cos\left((y - \hat{a}) \frac{k\pi}{\hat{b} - \hat{a}} \right) \mathrm{d}y.$$

- The efficient computation of $\phi_{Y_M}(u)$ is crucial for the performance of the whole procedure (specially, one-step case).
- The inversion of $F_{\log \hat{Y} | \log \sigma(s)}$ is relatively expensive (unafforable in the multi-step case).

Copula-based simulation of $\int_s^t \sigma^2(z)\mathrm{d}z | \sigma(t), \sigma(s)$

- In order to apply copulas, we need (logarithmic domain):
  - $F_{\log \hat{Y} | \log \sigma(s)}$.
  - $F_{\log \sigma(t) | \log \sigma(s)}$.
  - Correlation between $\log Y(s,t)$ and $\log \sigma(t)$.
- The distribution of $\log \sigma(t) | \log \sigma(s) = z$ is

$$\mathcal{N}\left(\mu_{\log \sigma(t)} + \mathcal{P}_{\log \sigma(t), \log \sigma(s)} \frac{s_{\log \sigma(t)}}{s_{\log \sigma(s)}} (z - \mu_{\log \sigma(t)}), s_{\log \sigma(t)} \sqrt{1 - \mathcal{P}^2_{\log \sigma(t), \log \sigma(s)}}\right),$$

  where all the quantities are known.
- Approximated Pearson's correlation coefficient:

$$\mathcal{P}_{\log Y, \log \sigma(t)} \approx \frac{t^2 - s^2}{2\sqrt{\left(\frac{1}{3}t^4 + \frac{2}{3}ts^3 - t^2 s^2\right)}}.$$

- For some copulas, like Archimedean, Kendall's τ is required:

$$\mathcal{P} = \sin\left(\frac{\pi}{2}\tau\right).$$

Sampling $\int_s^t \sigma^2(z)\mathrm{d}z|\sigma(t), \sigma(s)$: Steps

1. Determine $F_{\log\sigma(t)|\log\sigma(s)}$ and $F_{\log\hat{Y}|\log\sigma(s)}$.
2. Determine the correlation between $\log Y(s,t)$ and $\log\sigma(t)$.
3. Generate correlated uniform samples, $U_{\log\sigma(t)|\log\sigma(s)}$ and $U_{\log\hat{Y}|\log\sigma(s)}$ by means of copula.
4. From $U_{\log\sigma(t)|\log\sigma(s)}$ and $U_{\log\hat{Y}|\log\sigma(s)}$ invert original marginal distributions.
5. The samples of $\sigma(t)|\sigma(s)$ and $Y(s,t) = \int_s^t \sigma^2(z)\mathrm{d}z|\sigma(t), \sigma(s)$ are obtained by taking exponentials.

One time-step simulation of the SABR model

- $s = 0$ and $t = T$, with T the maturity time.
- The use is restricted to price European options up to $T = 2$.
- $\log \sigma(s)$ becomes constant.
- $F_{\log \sigma(t) | \log \sigma(s)}$ and $F_{\log \hat{Y} | \log \sigma(s)}$ turn into $F_{\log \sigma(T)}$ and $F_{\log \hat{Y}(T)}$.
- The computation of $\phi_{\log \hat{Y}(T)}$ is much simpler and very fast.
- The approximated Pearson's coefficient results in a constant value:

$$\mathcal{P}_{\log Y(T), \log \sigma(T)} \approx \frac{T^2}{2\sqrt{\frac{1}{3}T^4}} = \frac{\sqrt{3}}{2}.$$

Approximated correlation

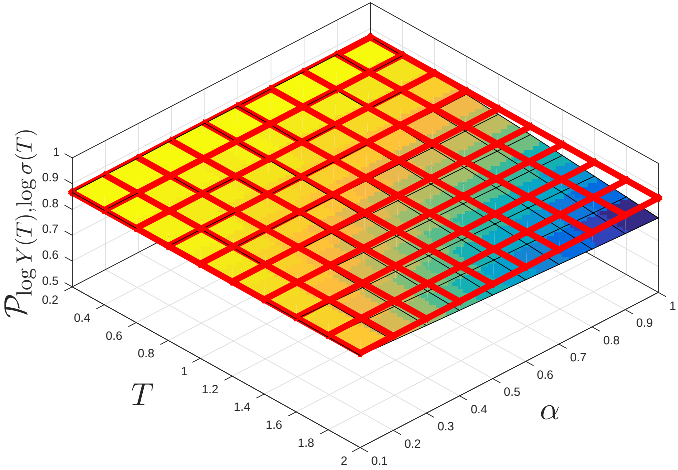


Figure: Pearson's coefficient: Empirical (surface) vs. approximation (red grid).

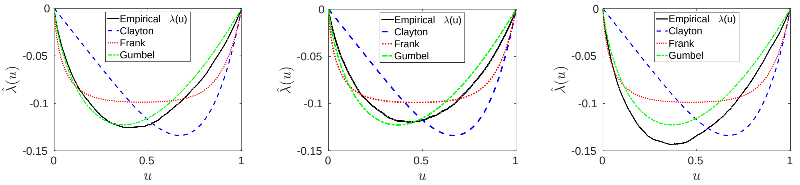
Copula analysis

- Based on the one-step simulation, a copula analysis is carried out.
- Gaussian, Student **t** and Archimedean (Clayton, Frank and Gumbel).
- A *goodness-of-fit (GOF)* for copulas needs to be evaluated.
- Archimedean: graphic GOF based on Kendall's processes.
- Generic GOF based on the so-called *Deheuvels or empirical copula*.

	S_0	σ_0	α	β	ρ	T
Set I	1.0	0.5	0.4	0.7	0.0	2
Set II	0.05	0.1	0.4	0.0	-0.8	0.5
Set III	0.04	0.4	0.8	1.0	-0.5	2

Table: Data sets.

GOF - Archimedean



(a) Set I. (b) Set II. (c) Set III.

Figure: Archimedean GOF test: $\hat{\lambda}(u)$ vs. empirical $\lambda(u)$.

	Clayton	Frank	Gumbel
Set I	1.3469×10^{-3}	2.9909×10^{-4}	5.1723×10^{-5}
Set II	1.0885×10^{-3}	2.1249×10^{-4}	8.4834×10^{-5}
Set III	2.1151×10^{-3}	7.5271×10^{-4}	2.6664×10^{-4}

Table: MSE of $\hat{\lambda}(u) - \lambda(u)$.

Generic GOF

	Gaussian	Student $\mathbf{t}$ $(\nu = 5)$	Gumbel
Set I	5.0323×10^{-3}	5.0242×10^{-3}	3.8063×10^{-3}
Set II	3.1049×10^{-3}	3.0659×10^{-3}	4.5703×10^{-3}
Set III	5.9439×10^{-3}	6.0041×10^{-3}	4.3210×10^{-3}

Table: Generic GOF: D_2.

- The three copulas perform very similarly.
- For longer maturities: Gumbel performs better.
- The Student **t** copula is discarded: very similar to the Gaussian copula and the calibration of the ν parameter adds extra complexity.
- As a general strategy, the Gumbel copula is the most robust choice.
- With short maturities, the Gaussian copula may be a satisfactory alternative.

Pricing European options

- The strike values X_i are chosen following the expression:

$$X_i(T) = f(0)\exp(0.1 \times T \times \delta_i),$$
$$\delta_i = -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5.$$

- Forward asset, $f(T)$: Bin Chen's enhanced inversion [2].
- Convergence and execution time in term of number of samples, n:

	$n = 1000$	$n = 10000$	$n = 100000$	$n = 1000000$
	Gaussian (Set I, X_1)			
Error	519.58	132.39	37.42	16.23
Time	0.3386	0.3440	0.3857	0.5733
	Gumbel (Set I, X_1)			
Error	151.44	−123.76	34.14	11.59
Time	0.3492	0.3561	0.3874	0.6663

Table: Convergence in n: error (basis points) and time (*sec.*).

Pricing European options - Implied volatilities

Strikes	X_1	X_2	X_3	X_4	X_5	X_6	X_7
	Set I (Reference: Antonov [1])						
Hagan	55.07	52.34	50.08	N/A	47.04	46.26	45.97
MC	23.50	21.41	19.38	N/A	16.59	15.58	14.63
Gaussian	16.23	20.79	24.95	N/A	33.40	37.03	40.72
Gumbel	11.59	15.57	19.12	N/A	25.41	28.66	31.79
	Set II (Reference: Korn [7])						
Hagan	−558.82	−492.37	−432.11	−377.47	−327.92	−282.98	−242.22
MC	5.30	6.50	7.85	9.32	10.82	12.25	13.66
Gaussian	9.93	9.98	10.02	10.20	10.57	10.73	11.04
Gumbel	−9.93	−9.38	−8.94	−8.35	−7.69	−6.83	−5.79
	Set III (Reference: MC Milstein)						
Hagan	287.05	252.91	220.39	190.36	163.87	141.88	126.39
Gaussian	16.10	16.76	16.62	15.22	13.85	12.29	10.67
Gumbel	6.99	3.79	0.67	−2.27	−5.57	−9.79	−14.06

Table: Implied volatility: errors in basis points.

- One-step SABR simulation is a fast alternative to Hagan formula.
- Overcomes the known issues, like low strikes and high volatilities.
- For long maturities: multiple time-step version.

Multiple time-step simulation of the SABR model

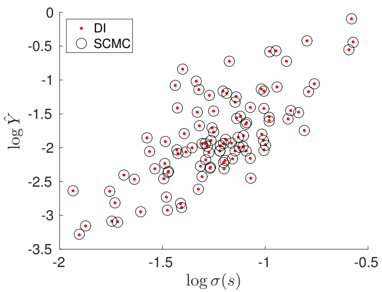
- In intermediate steps, $\phi_{\log \hat{Y}|\log \sigma(s)}$ becomes "stochastic".
- $f_{\log \hat{Y}|\log \sigma(s)}$ needs to be computed for each sample of $\log \sigma(s)$.
- Consequently, the inversion of $F_{\log \hat{Y}|\log \sigma(s)}$ is unafforable ($n \uparrow\uparrow$).
- Solution: *Stochastic Collocation Monte Carlo* (SCMC) sampler [4].

$$y_n|v_n \approx g_{L_{\hat{Y}},L_\sigma}(x_n) = \sum_{i=1}^{L_{\hat{Y}}} \sum_{j=1}^{L_\sigma} F^{-1}_{\log \hat{Y}|\log \sigma(s)=v_j}(F_X(x_i))\ell_i(x_n)\ell_j(v_n),$$

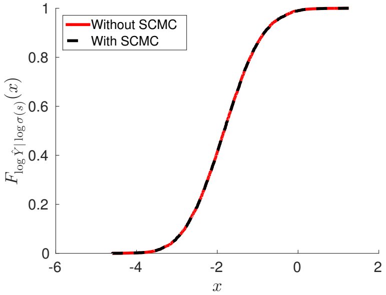
where x_n are the samples from the *cheap variable*, X, and v_n the given samples of $\log \sigma(s)$. x_i and v_j are the *collocation points* of X and $\log \sigma(s)$, respectively. ℓ_i and ℓ_j are the Lagrange polynomials defined by

$$\ell_i(x_n) = \prod_{k=1,k\neq i}^{L_{\hat{Y}}} \frac{x_n - x_k}{x_i - x_k}, \quad \ell_j(v_n) = \prod_{k=1,k\neq j}^{L_\sigma} \frac{v_n - v_k}{v_i - v_k}.$$

Application of 2D SCMC to $F_{\log \hat{Y} | \log \sigma(s)}$



(a) $\log \hat{Y} | \log \sigma(s)$ - Direct inversion vs. SCMC.



(b) $F_{\log \hat{Y} | \log \sigma(s)}(x)$.

Samples	Without SCMC	With SCMC		
		$L_{\hat{Y}} = L_\sigma = 3$	$L_{\hat{Y}} = L_\sigma = 7$	$L_{\hat{Y}} = L_\sigma = 11$
100	1.0695	0.0449	0.0466	0.0660
10000	16.3483	0.0518	0.0588	0.0798
1000000	1624.3019	0.2648	0.5882	1.0940

Table: SCMC time.

Multi-step SABR simulation - Pricing

- Data sets.

	S_0	σ_0	α	β	ρ	T
Set I	1	0.3	0.5	1.0	-0.8	5
Set II	0.5	0.5	0.4	0.5	0.0	2
Set III	0.035	0.01	0.5	0.0	0.0	30

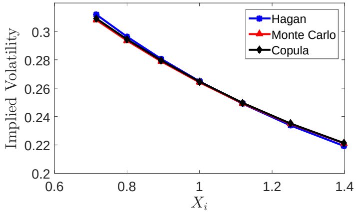
- Convergence in term of number of time-steps, m (Set II).

Strikes	X_1	X_2	X_3	X_4	X_5	X_6	X_7
Antonov[1]	75.51%	74.18%	72.90%	N/A	70.47%	69.32%	68.22%
Copula ($m = T/2$)	72.43%	71.45%	70.49%	69.55%	68.63%	67.74%	66.88%
Diff.(bp)	-307.56	-272.86	-240.61	N/A	-183.50	-158.05	-134.38
Copula ($m = T$)	74.65%	73.43%	72.25%	71.11%	70.00%	68.93%	67.91%
Diff.(bp)	-85.84	-74.88	-64.74	N/A	-46.65	-38.66	-31.33
Copula ($m = 2T$)	75.43%	74.14%	72.89%	71.68%	70.51%	69.39%	68.31%
Diff.(bp)	-8.00	-4.70	-1.39	N/A	4.13	6.44	8.58
Copula ($m = 12T$)	75.55%	74.22%	72.93%	71.68%	70.48%	69.33%	68.23%
Diff.(bp)	4.45	3.70	2.82	N/A	1.58	0.92	0.36

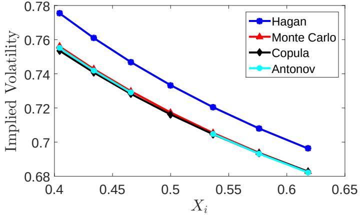
Table: Implied volatility: Antonov vs. Copula.

Pricing - Implied volatilities

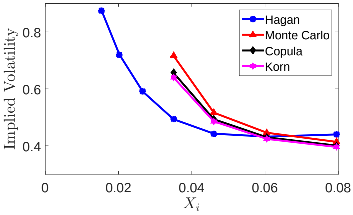
- $n = 1000000$ and $m = 2T$:



(c) Implied volatility set I.



(d) Implied volatility set II.



(e) Implied volatility set III.

Conclusions

- We propose an efficient SABR simulation based on Fourier and copula techniques.
- The one-step SABR is a fast alternative to Hagan formula for short maturities.
- Overcomes the known issues of Hagan's expression.
- When long maturities are considered, multi-step version.
- High accuracy with very few number of time-steps.

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Questions

Thank you for your attention