Quantum computing for computational finance

Review of promising algorithms for pricing and Var

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Premia meetting 2024

June 5, 2024

Motivation

- Quantum computers could bring unparalleled competitive advantage to financial companies in areas like portfolio optimisation, option pricing, quantitative risk management or Machine Learning models.
- Quantum computers are able to handle exponentially growing (in qubits) Hilbert spaces.
- Thus, quantum computing becomes an attractive framework for calculations over large multi-dimensional domains.
- Quantum algorithms could potentially overcome their classical counterparts in dealing with combinatorial explosions and the curse of dimensionality.
- However, bringing this to practice encounters several bottlenecks, especially with the current or near-term quantum technologies (NISQ).

Disclaimer

- Quantum computing literature is experiencing an explosion: This presentation incorporates only a few of the current trends.
- The selection of the addressed topics reflects only my view (interests) within the vast scope of the computational finance field.
- Then, many important topics are not addressed here: optimization, time series, blockchain, cryptography, etc.
- There might be inconsistencies or certain abuse in the (mathematical and/or quantum) notation. In some cases, that is intentional, for the sake of clarity. In others...sorry in advance!

Outline

Quantum Computing basics

Promising Quantum Algorithms for pricing (and risk measures)

- Quantum Monte Carlo
- Quantum PDE solvers

Quantum Machine/Deep Learning

Discussion

Quantum Computing basics

Quantum Computing basics (I)

- The basic unit of information is the *qubit* (alternatively to the *bit*).
- A qubit is represented by a (column) vector:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

with the *amplitudes* $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.

- Basis states:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- $\{|0\rangle, |1\rangle\}$ is a computational basis for a quantum state:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

- When *measuring* the state:
  - get 0 with probability $|\alpha|^2$
  - get 1 with probability $|\beta|^2$
- So, *measurement* $\equiv$ *distribution sampling*

Promising Quantum Algorithms for pricing (and risk measures)

Quantum Monte Carlo

Quantum Monte Carlo (QMC)

The quantum-accelerated Monte Carlo could potentially/theoretically provide a quadratic speedup for option pricing and risk measures calculation [Gómez et al., 2022].

How? Quantum Amplitude Estimation.

Monte Carlo methods in finance can be informally defined as

$$\frac{1}{M}\sum_{i=0}^{M-1} f(X_i) \approx \mathbb{E}[f(X)] = \int f(x)p(x)dx \approx \sum_{j=0}^{N-1} f(x_j)p(x_j)$$

where $p(x)$ is a density function.

Analogously, Quantum Monte Carlo (QMC) assumes a state of the form

$$|\psi\rangle = |0\rangle \otimes \sum_{j=0}^{N-1} f(x_j)p(x_j)\,|j\rangle + |1\rangle \otimes \sum_{j=0}^{N-1} \sqrt{1 - f^2(x_j)p(x_j)}\,|j\rangle$$

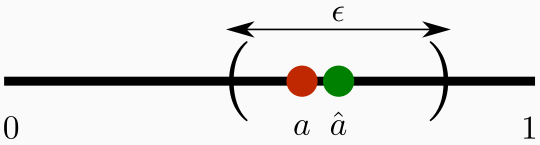
where the quantity of interest is encoded in the state's amplitude.

QMC: Quantum Amplitude Estimation

Given a state:

$$|\psi\rangle = a\,|\phi\rangle + \sqrt{1-a^2}\,|\phi^{\perp}\rangle\,,$$

Quantum Amplitude Estimation (QAE) is an algorithm which gives an estimation $\hat{a} \pm \frac{\epsilon}{2}$ of the amplitude a.



This technique promises to obtain a **quadratic speedup over its classical counterpart**.

To achieve so, it relies on two main subroutines:

- Grover (search) amplification.
- Quantum Phase Estimation.

QMC: Quantum Amplitude Estimation (circuit)

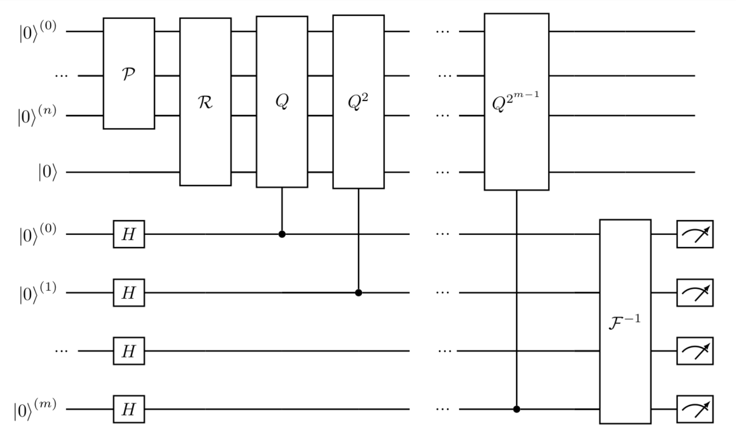


Figure 1: Quantum Amplitude Estimation

QMC: Quantum Amplitude Estimation (convergence)

Theorem (Mean estimation for $[0,1]$ bounded functions [Montanaro, 2015])
Let there be given a quantum circuit $\mathcal{P}$ on n qubits. Let $v(\mathcal{P})$ be the random variable that maps to $v(x) \in [0,1]$ when the bit string x is measured as the output of $\mathcal{P}$. Let $\mathcal{R}$ be defined as

$$\mathcal{R}\,|x\rangle\,|0\rangle = |x\rangle\left(\sqrt{1-v(x)}\,|0\rangle - \sqrt{v(x)}\,|1\rangle\right).$$

Let $|\mathcal{X}\rangle$ be defined as $|\mathcal{X}\rangle = \mathcal{R}(\mathcal{P}\otimes\mathcal{I}_2)\left|0^{n+1}\right\rangle$. Set $\mathcal{U} = \mathcal{I}_{2^{n+1}} - 2\,|\mathcal{X}\rangle\,\langle\mathcal{X}|$. There exists a quantum algorithm that uses $\mathcal{O}(\log 1/\delta)$ copies of the state $\mathcal{X}$, uses $\mathcal{U}$ for a number of times proportional to $\mathcal{O}(m\log 1/\delta)$ and outputs an estimate $\hat{\mu}$ such that

$$|\hat{\mu} - \mathbb{E}[v(\mathcal{P})]| \leq C\left(\frac{\sqrt{\mathbb{E}[v(\mathcal{P})]}}{m} + \frac{1}{m^2}\right),$$

with probability at least $1-\delta$, where C is a universal constant. In particular, for any fixed $\delta > 0$ and any ϵ such that $0 < \epsilon \leq 1$, to produce an estimate $\hat{\mu}$ such that, with probability at least $1-\delta$, $|\hat{\mu} - \mathbb{E}[v(\mathcal{P})]| \leq \epsilon\mathbb{E}[v(\mathcal{P})]$, it suffices to take $m = \mathcal{O}\left((\epsilon\mathbb{E}[v(\mathcal{P})])^{-1}\right)$. To achieve $|\hat{\mu} - \mathbb{E}[v(\mathcal{P})]| \leq \epsilon$ with probability at least $1-\delta$, it suffices to take $m = \mathcal{O}\left(\epsilon^{-1}\right)$.

QMC: variations on QAE

Plain QAE is not feasible in NISQ era, due to the use of a Quantum Fourier Transform (QFT).

Algorithm	Performance
Monte Carlo	$N_{\mathcal{A}}^{MC} \sim \mathcal{O}\left(\frac{1}{\epsilon_p^2}\right)$
QPE[Brassard et al., 2002]	$N_{\mathcal{A}}^{QPE} \sim \mathcal{O}\left(\frac{1}{\epsilon_p}\right)$
MLAE-LIS[Suzuki et al., 2020]	$N_{\mathcal{A}}^{LIS} \sim \mathcal{O}\left(\epsilon_p^{-4/3}\right)$
MLAE-EIS[Suzuki et al., 2020]	$N_{\mathcal{A}}^{EIS} \sim \mathcal{O}\left(\frac{1}{\epsilon_p}\right)$
PLAE[Giurgica-Tiron et al., 2020]	$N_{\mathcal{A}}^{PLAE} \sim \mathcal{O}\left(\frac{1}{\epsilon_p^{1+\beta}}\right)$, $d \sim \mathcal{O}\left(\frac{1}{\epsilon_p^{1-\beta}}\right)$
Improved MLAE[Callison and Browne, 2022]	$N_{\mathcal{A}}^{imp\ EIS} \sim \mathcal{O}\left(\frac{1}{\epsilon_p}\frac{1}{d}\log\left(\frac{1}{\gamma}\right)\right)$, $d = 2^{q-2}$
IQAE [Grinko et al., 2021]	$N_{\mathcal{A}}^{IQAE} < \frac{50}{\epsilon_p}\log\left(\frac{2}{\gamma}\log_2\frac{\pi}{4\epsilon_p}\right)$
mIQAE[Fukuzawa et al., 2023]	$N_{\mathcal{A}}^{mIQAE} < \frac{123}{\epsilon_p}\log\frac{6}{\gamma}$
QCoin [Abrams and Williams, 1999]	$N_{\mathcal{A}}^{QCoin} \sim \mathcal{O}\left(\frac{1}{a}\frac{1}{\epsilon_p}\log\frac{1}{\gamma}\right)$, $k \geq 2,\ 1 \geq q \geq (k-1)$
QoPrime [Giurgica-Tiron et al., 2020]	$N_{\mathcal{A}}^{QoPrime} < C\lceil\frac{k}{q}\rceil\frac{1}{\epsilon_p^{1+q/k}}\log\left(\frac{4}{\gamma}\lceil\frac{k}{q}\rceil\right)$, $d \sim \mathcal{O}\left(\frac{1}{\epsilon_p^{1-q/k}}\right)$
FasterAE [Nakaji, 2020]	$N_{\mathcal{A}}^{fasterAE} < \frac{4.1\cdot10^3}{\epsilon_p}\log\left(\frac{4}{\gamma}\log_2\left(\frac{2\pi}{3\epsilon_p}\right)\right)$
AdaptiveAE [Zhao et al., 2022]	$N_{\mathcal{A}}^{adaptiveAE} < \mathcal{O}\left(\frac{1}{\epsilon_p}\log\left(\frac{\pi^2(T+1)}{3\gamma}\right)\right)$, $T = \lceil\frac{\log\frac{\pi}{K\epsilon_p}}{\log K}\rceil$
RQAE [Manzano et al., 2023b]	$N_{\mathcal{A}}^{RQAE} < \frac{C_1(q)}{\epsilon_a}\log\left[\frac{3.3}{\gamma}\log_q\left(\frac{C_2(q)}{\epsilon_a}\right)\right]$
mRQAE [Ferro and Manzano, 2024]	$N_{\mathcal{A}}^{MRQAE} < \frac{C_1(q)}{\epsilon_a}\log\left[\frac{C_2(q)}{\gamma}\right]$

Quantum PDE solvers

Quantum PDE solvers

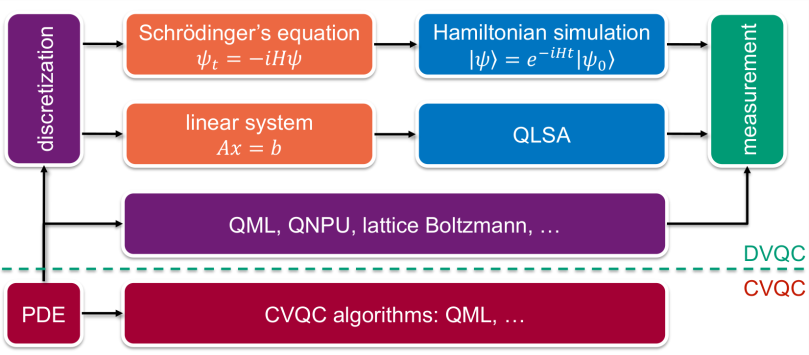


Figure 2: Classification of quantum PDE solvers.

Quantum approaches for Black-Scholes PDE

Some financial PDEs can be mapped into the propagation governed by a Hamiltonian [Gonzalez-Conde et al., 2021, Fontanela et al., 2021].

Applying the change of variable $S = e^x$ on the Black-Scholes eq.,

$$\frac{\partial V}{\partial t} + \left(\mu - \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} - \mu V = 0 \ ,$$

which can be written as a Schrödinger-like equation,

$$\frac{\partial V}{\partial t} = -i\hat{H}_{\text{BS}}\, V \ ,$$

where

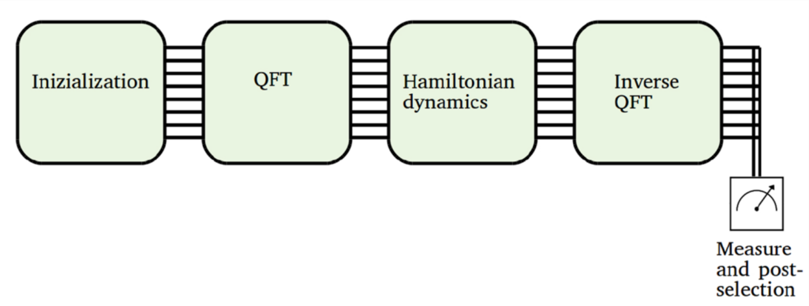
$$\hat{H}_{\text{BS}} = i\frac{\sigma^2}{2}\hat{p}^2 - \left(\frac{\sigma^2}{2} - \mu\right)\hat{p} + i\mu\mathbb{I} \ , \quad \text{with} \quad \hat{p} = -i\frac{\partial}{\partial x} \ .$$

The Hamiltonian $\hat{H}_{\text{BS}}$ is *not* Hermitian.

Therefore, the associated evolution operator $\hat{U}(t, t_0) = e^{-i\hat{H}_{\text{BS}}(t-t_0)}$ is not unitary.

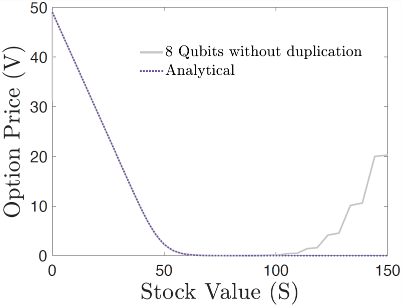
PDEs: Real-time propagation

- To implement $\hat{U}(t, t_0)$, consider an enlarged system, i.e. a doubled unitary operator [Gonzalez-Conde et al., 2021].
- Require of adding an auxiliary qubit.
- $\hat{H}_{\text{BS}}$ is diagonal in momentum space → diagonal operator → QFT (and Inverse QFT) → exponential speedup.
- But, an overall exponential speedup requires efficient loading of the model and payoff function.
- Again, QFT (IQFT) is gate-wise demanding (incompatible NISQ).

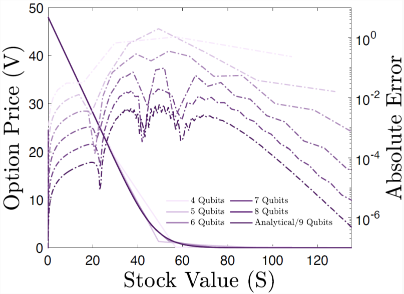


PDEs: Real-time propagation (solution)

- The algorithm achieves a high degree of agreement in a fault-tolerant quantum computer...
- ...but with a 60% success probability in the measurement and post-selection (depending on the financial parameters).
- Not tested in a real NISQ quantum system.



(a) Boundary error without duplication.



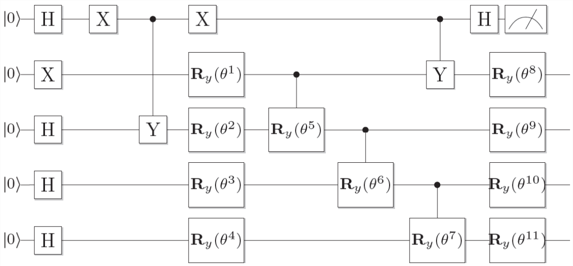
(b) Convergence in qubits (point).

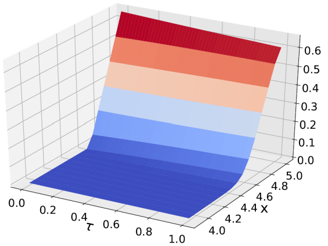
PDEs: Imaginary-time propagation

- Additional change of variable $\tau = \sigma^2(T - t)$ and transformation $v(x, \tau) = \exp(-ax - b\tau)V(t, s)$, with suitable constants a and b,
$$\frac{\partial v}{\partial \tau} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} .$$
- Using the Wick rotation $\tilde{\tau} = -i\tau$ (real time to imaginary time), the heat equation turns into a Schrödinger-like equation,
$$\frac{\partial v}{\partial \tilde{\tau}} = -\hat{H}_{\mathrm{HE}}\, v ,$$
where
$$\hat{H}_{\mathrm{HE}} = -\frac{i}{2}\hat{q}^2 , \quad \text{with} \quad \hat{q} = -i\frac{\partial}{\partial x}.$$
- This leads to a purely anti-Hermitian Hamiltonian operator.
- Imaginary-time propagation transforms oscillations into dampings.
- Problem of finding the ground state of quantum systems, well investigated in condensed matter physics and chemistry.
- The imaginary time evolution operator is approximated by an ansatz circuit in [Fontanela et al., 2021].

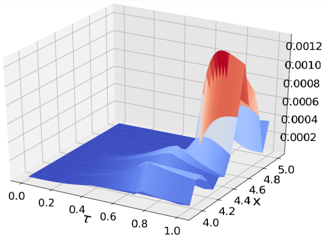
PDEs: Imaginary-time propagation (solution)

- The solution is retrieved by a hybrid quantum-classical algorithm:





(c) Prices of European option.



(d) Errors of European option.

Quantum Machine/Deep Learning

Quantum Machine/Deep Learning

- (Quantum) Principal Component Analysis:
  - Eigenvalues by QPE [Nielsen and Chuang, 2001].
  - Covariance matrix $\rightarrow$ density matrix (QPCA) [Lloyd et al., 2014, Abhijith et al., 2020].
- (Quantum) Regression:
  - Solving linear systems by the HHL algorithm [Wiebe et al., 2012].
  - Quantum Kernel Estimation [Egger et al., 2020].
  - Quantum regression with Gaussian processes [Zhao et al., 2019].
- Hybrid classical-quantum deep learning:
  - Training in QC (quantum annealing) [Adachi and Henderson, 2015].
  - Quantum-enhanced reinforcement learning [Saggio et al., 2021].
  - Quantum GANs [Nakaji et al., 2021].
  - Boltzman machines $\rightarrow$ Born machines [Vinci et al., 2020, Alcazar et al., 2020].
- Full Quantum Neural Network (QNN):
  - ANNs based on the principles of quantum mechanics [Kak, 1995].
  - How to train QNN? See [Beer et al., 2020, Coyle et al., 2021].
- Promising approach: *Parametrized Quantum Circuits*

Parametrized Quantum Circuits (PQCs)

- Also known as *variational circuits* or *quantum circuit learning*.
- First theoretical results on *accessibility*, *expressivity* and *universality*.
- Circuits with both fixed and adjustable ("parametrized") gates.
- The training is carried out by a classical optimiser.
- Each layer composed by a trainable circuit block $W_i(\theta)$ and a data-encoding block $S(x)$:

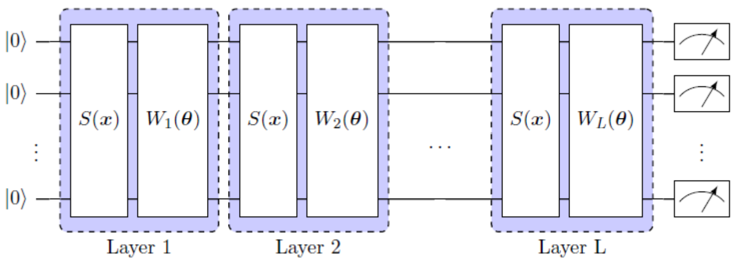


Figure 3: Parametrized Quantum Circuit.

PQCs: trigonometric series

- A PQC model can be written as a generalized trigonometric series:

$$\mathbb{E}[M] = \langle 0| U^{\dagger}(\boldsymbol{x};\boldsymbol{\theta}) M U(\boldsymbol{x};\boldsymbol{\theta}) |0\rangle = f(\boldsymbol{x};\boldsymbol{\theta}) = \sum_{\boldsymbol{\omega}\in\Omega} c_{\boldsymbol{\omega}}(\theta) e^{i\boldsymbol{\omega}\boldsymbol{x}},$$

  where M is an observable, $U(\boldsymbol{x};\boldsymbol{\theta})$ is a quantum circuit that depends on inputs $\boldsymbol{x} = (x_0, x_1, ..., x_N)$ and the parameters $\boldsymbol{\theta} = (\theta_0, \theta_1, ..., \theta_T)$.
- Accessibility: with $\Omega \subset \mathbb{Z}^N \rightarrow$ (partial) Fourier series!
- The coefficients $c_{\boldsymbol{\omega}}$ determine the expressivity (how the accessible functions can be combined).
- But the expressivity is also limited by the data encoding strategy.
- Universality: the Fourier series formalism allows to study quantum models using the results in Fourier analysis (see [Schuld et al., 2021] and [Manzano et al., 2023a]).

PQCs: Universality results (I)

Definition
Let $U(\boldsymbol{x};\boldsymbol{\theta})$ be modelled as a unitary such that (1 layer):

$$U(\boldsymbol{\theta},\boldsymbol{x}) = W^{(2)}(\boldsymbol{\theta}^{(2)})S(\boldsymbol{x})W^{(1)}(\boldsymbol{\theta}^{(1)}),$$

and

$$S(\boldsymbol{x}) = e^{-x_1 H} \otimes \cdots \otimes e^{-x_N H} =: S_H(\boldsymbol{x})$$

where H is a particular Hamiltonian.

Definition
Let $\{H_m | m \in \mathbb{N}\}$ be a Hamiltonian family where H_m acts on m subsystems of dimension d. Such a Hamiltonian family gives rise to a family of models $\{f_m\}$ in the following way:

$$f_m(\boldsymbol{x}) = \langle\Gamma| S^{\dagger}_{H_m}(\boldsymbol{x}) M S_{H_m}(\boldsymbol{x}) |\Gamma\rangle \ . \tag{1}$$

with $|\Gamma\rangle := W^{(1)}(\boldsymbol{\theta}^{(1)}) |0\rangle$.

PQCs: Universality results (II)

Theorem (Convergence in L^2 [Schuld et al., 2021])
Let $\{H_m\}$ be a universal Hamiltonian family, and $\{f_m\}$ the associated quantum model family, defined via (1). For all functions $f^* \in L^2\left([0, 2\pi]^N\right)$, and for all $\epsilon > 0$, there exists some $m' \in \mathbb{N}$, some state $|\Gamma\rangle \in \mathbb{C}^{m'}$ and some observable M such that

$$\|f_{m'} - f^*\|_{L^2} < \epsilon.$$

Theorem (Convergence in L^p [Manzano et al., 2023a])
Let $\{H_m\}$ be a universal Hamiltonian family, and $\{f_m\}$ the associated quantum model family, defined via (1). For all functions $f^* \in L^p\left([0, 2\pi]^N\right)$ where $1 \leq p < \infty$, and for all $\epsilon > 0$, there exists some $m' \in \mathbb{N}$, some state $|\Gamma\rangle \in \mathbb{C}^{m'}$, and some observable M such that:

$$\|f_{m'} - f^*\|_{L^p} < \epsilon.$$

PQCs: Universality results (and III)

Theorem (Convergence in C^0 [Manzano et al., 2023a])
Let $\{H_m\}$ be a universal Hamiltonian family, and $\{f_m\}$ the associated quantum model family, defined via (1). For all functions $f^* \in C^0(U)$ where U is compactly contained in the closed cube $[0, 2\pi]^N$, and for all $\epsilon > 0$, there exists some $m' \in \mathbb{N}$, some state $|\Gamma\rangle \in \mathbb{C}^{m'}$, and some observable M such that $f_{m'}$ converges uniformly to f^*:

$$\|f_{m'} - f^*\|_{C^0} < \epsilon,$$

with

$$\|f_{m'} - f^*\|_{C^0} := \sup_{\mathbf{x} \in [0,2\pi]^N} \|f_{m'}(\mathbf{x}) - f^*(\mathbf{x})\| \; .$$

PQCs for Black-Scholes distribution

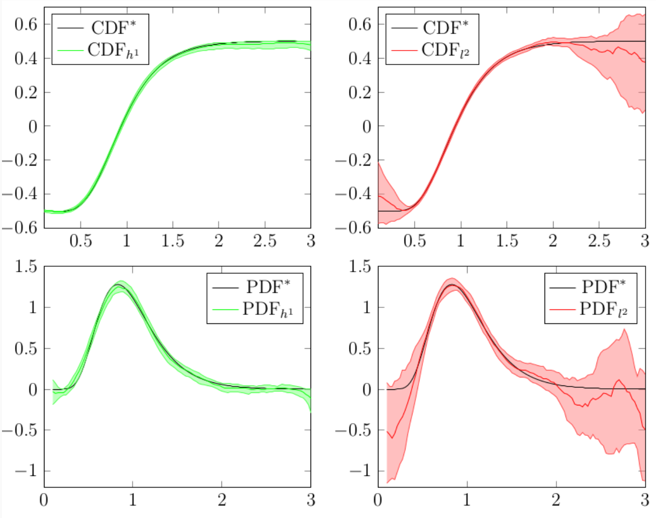


Figure 4: PQC approximating the Black-Scholes distribution, using the two different empirical risk functions associated to L^2 and C^0 convergence results, respectively.

Discussion

Discussion on Quantum Monte Carlo

Is the Quantum Monte Carlo what we (computational finance community) expect?

In [Stamatopoulos et al., 2020] they divide the routine for computing the price of a plain vanilla in three steps:

They **promise a quadratic speedup over classical Monte Carlo**:

"This represents a theoretical quadratic speed-up compared to classical Monte Carlo methods."

Classical Monte Carlo vs Quantum Monte Carlo

When claiming a **"quadratic"** speedup of the QMC over the Classical, **what are they comparing?**

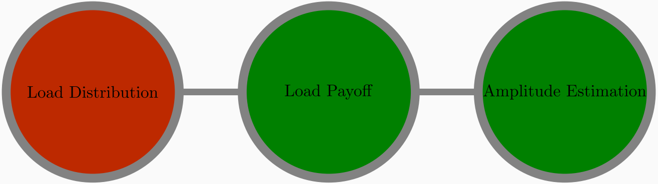
Steps involved in Classical and Quantum Monte Carlo :

Quantum Monte Carlo	**Classical Monte Carlo**
Load Distribution	Load parameters... Simulate the paths
Load Payoff	Compute payoff
Amplitude Estimation	Sum over paths Print the results

"In most of the existing literature on option pricing for equities using quantum computers... an SDE is tacitly solved... Once this SDE is solved... the pricing of a particular security begins by applying QAE.[Alghassi et al., 2021]"

Bottleneck

The bottleneck in Classical Monte Carlo is in simulating paths.
Analogously, the bottleneck in the quantum algorithm is in the loading/simulation/computation of the distribution.



The quantum advantage might disappear when taking into account the cost of simulation:

"Although preparing such states is in principle always possible for reasonable stochastic processes, efficient realization of this method demands a careful analysis and may not always result in a practical quantum advantage." (see [Alghassi et al., 2021])

Quantum Monte Carlo simulation

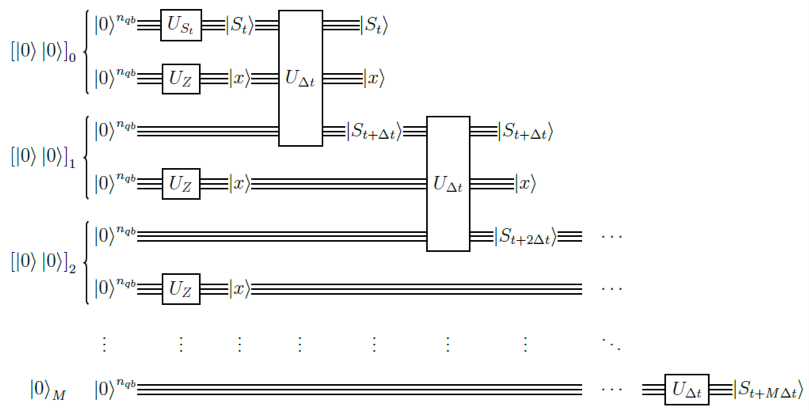


Figure 5: Quantum Monte Carlo simulation.

Quantum Monte Carlo simulation

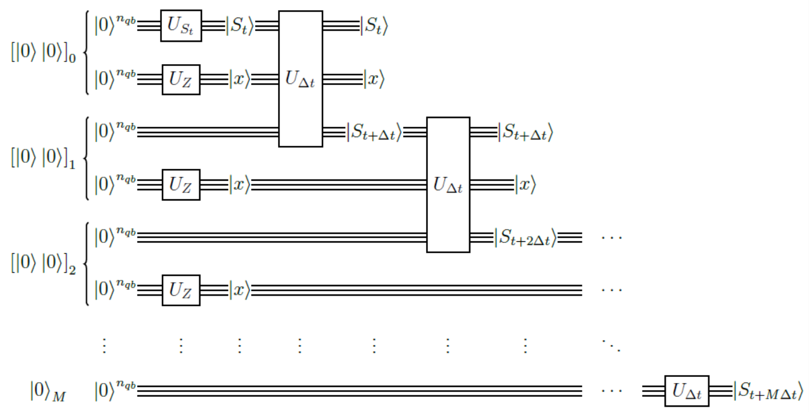


Figure 5: Quantum Monte Carlo simulation.

In single precision and $M = 12$: ~ 800 (logical) qubits!

Ranking of quantum computers in number of qubits

Circuit-based quantum processors [edit]

These QPUs are based on the quantum circuit and quantum logic gate-based model of computing.

Manufacturer	Name/codename designation	Architecture	Layout	Fidelity (%)	Qubits (physical)	Release date	Quantum volume
Atom Computing	N/A	Neutral atoms in optical lattices			1180[6][7]	October 2023	
IBM	IBM Condor[16][6]	Superconducting	N/A	N/A	1121[15]	December 2023	
CAS	Xiaohong[64]	Superconducting	N/A	N/A	504[64]	2024	
IBM	IBM Osprey[6][7]	Superconducting	N/A	N/A	433[15]	November 2022	
Xanadu	Borealis[62]	Photonics (Continuous-variable)	N/A	N/A	216[62]	2022[62]	
M Squared Lasers	Maxwell	Neutral atoms in optical lattices		99.5 (3-qubit gate), 99.1 (4-qubit gate)[31]	200[32]	November 2022	
IBM	IBM Heron[16][6]	Superconducting	N/A	N/A	133	December 2023	
IBM	IBM Eagle	Superconducting	N/A	N/A	127[15]	November 2021	
Atom Computing	Phoenix	Neutral atoms in optical lattices			100[5]	August 10, 2021	

Figure 6: Quantum computers with more than 100 (physical) qubits (05/06/2024).

Other challenges in algorithms for quantitative finance

- Data accessibility for Quantum Machine Learning models.
- Quantum-native function implementations (using unitary transforms).
- Information extraction from a quantum state:
  - QAE can be seen as an efficient information extraction routine
  - Post selection in PDE-Hamiltonian simulation algorithms?
- Rigorous proofs for:
  - Speedups (quantum advantage)
  - Estimation convergence
  - Circuits complexity (depth)
- Quantum volume (NISQ):
  - Intrinsic noise of the current quantum systems (the shallower the better)
  - Limited number of qubits (i.e. to represent floating-point numbers)
  - Others: coherence time, measurement errors, circuit compiler efficiency, etc.

Conclusions

- In recent years we have seen significant advances in quantum algorithms with application to financial mathematical problems.
- While this progress is very encouraging, further work will be required to prove that Quantum Computing can deliver real-world advantage.
- Especially if this advantage is to be delivered on NISQ technology with limitations to both the number of logical qubits and the depth of quantum circuits.
- Research into financial applications of quantum computing is accelerating with new ideas emerging at rapid pace...
- ...but important breakthroughs across the technology stack will be needed to make the approaches viable.
- Theory/software is ahead of practice/hardware!
- Plenty of room for contributions!

Merci!!

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Thanks to the support of Centre for Information and Communications Technology Research (CITIC). CITIC is funded by the Xunta de Galicia through the collaboration agreement between the Consellería de Cultura, Educación, Formación Profesional e Universidades and the Galician universities for the reinforcement of the research centres of the Galician University System (CIGUS).

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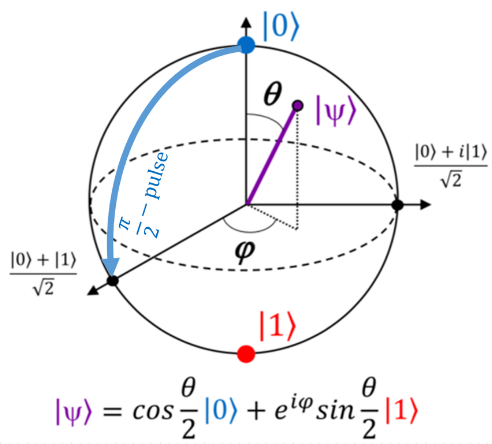
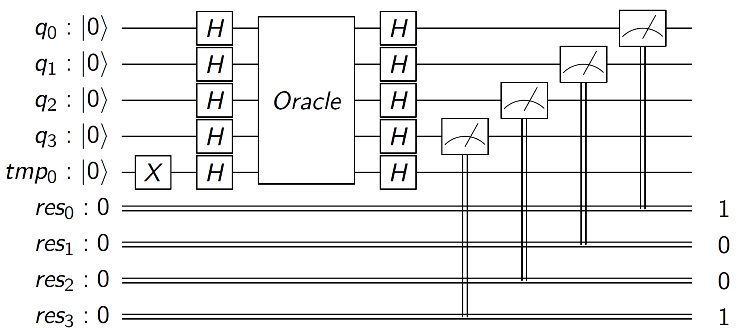
Quantum Computing basics (II)

- The Bloch sphere provides a representation of qubit state
- Measuring a qubit occurs along the Z axis, so it is irreversible and will collapse to either 0 or 1



Quantum Computing basics (and III)

- Each row represents a bit, either quantum or classical
- The operations are performed each qubit from left to right
- Measurement to extract the information



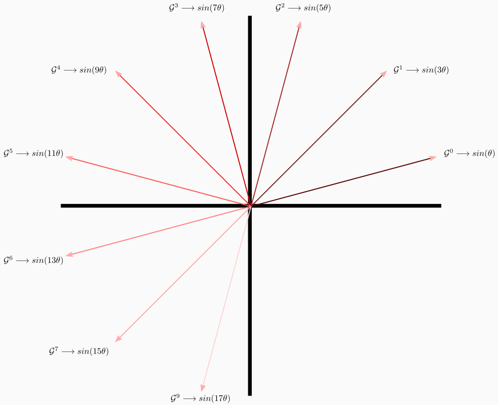
QMC: Grover Amplification

Given a state:

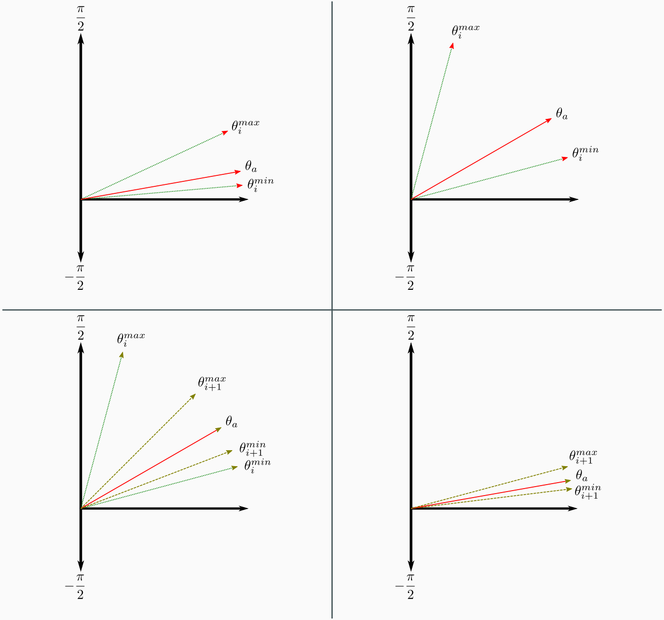
$$|\psi\rangle = \sin(\theta)\,|\phi\rangle + \cos(\theta)\,|\phi^{\perp}\rangle\,,$$

Grover operator performs the following transformation:

$$\mathcal{Q}^k\,|\psi\rangle = \sin((2k+1)\theta)\,|\phi\rangle + \cos((2k+1)\theta)\,|\phi^{\perp}\rangle\,.$$



QMC: Quantum Amplitude Estimation (graphically)



QMC: Risk measures

Find $\mathrm{VaR}_\alpha(X) = \inf\{x : \mathbb{P}[X \leq x]\} \geq 1 - \alpha\} = \inf\{x : F_X(x) \geq 1 - \alpha\}$:

$$f_J(x) = \begin{cases} 1 & \text{if } x \leq x_J \\ 0 & \text{otherwise} \end{cases}$$

Thus, the original QMC state becomes

$$|\psi\rangle = |0\rangle \otimes \sum_{j=J+1}^{N-1} p(x_j)\,|j\rangle + |1\rangle \otimes \sum_{j=0}^{J} \sqrt{p(x_j)}\,|j\rangle$$

A bisection search over J and measuring $|1\rangle$ gives the $x_{J_\alpha} \approx \mathrm{VaR}_\alpha(X)$

To estimate $\mathrm{CVaR}_\alpha(X)$, take $f(x) = \frac{x}{x_{J_\alpha}} f_{J_\alpha}(x)$, so

$$|\psi\rangle = |0\rangle \otimes \left(\sum_{j=J_\alpha+1}^{N-1} p(x_j)\,|j\rangle + \sum_{j=0}^{J_\alpha} \left(1 - \frac{x_j}{x_{J_\alpha}}\right) p(x_j)\,|j\rangle \right) + |1\rangle \otimes \sum_{j=0}^{J_\alpha} \sqrt{\frac{x_j}{x_{J_\alpha}} p(x_j)}\,|j$$

and measure $|1\rangle$. Then, $\mathrm{CVaR}_\alpha(X) \approx \frac{x_{J_\alpha}}{1-\alpha} \sum_{j=0}^{J_\alpha} \frac{x_j}{x_{J_\alpha}} p(x_j)$

QMC: Pricing

Using Y-rotations and a comparator (in K), we can construct:

$$|\psi\rangle = |0\rangle \otimes \sum_{x_j < K} \sqrt{p(x_j)}\, |j\rangle \left[\cos(g_0)\,|0\rangle + \sin(g_0)\,|1\rangle\right]$$

$$+ |1\rangle \otimes \sum_{x_j \geq K} \sqrt{p(x_j)}\, |j\rangle \left[\cos(g_0 + g(x_j))\,|0\rangle + \sin(g_0 + g(x_j))\,|1\rangle\right]$$

The probability of measuring the second *ancilla* (auxiliary) state $|1\rangle$ is:

$$P = \sum_{x_j < K} p(x_j) \sin^2(g_0) + \sum_{x_j \geq K} p(x_j) \sin^2(g_0 + g(x_j))$$

For a European call ($\max(0, x_j - K)$), set $g(x) = \frac{2c(x-K)}{x_{max}-K}$, $g_0 = \frac{\pi}{4} - c$.

Thus, using that $sin^2(cf(x) + \frac{\pi}{4}) = cf(x) + \frac{1}{2} + \mathcal{O}(c^3 f^3(x))$, we have

$$P \approx \sum_{x_j < K} p(x_j) \left(\frac{1}{2} - c\right) + \sum_{x_j \geq K} p(x_j) \left(\frac{2c(x_j - K)}{x_{max} - K} + \frac{1}{2} - c\right)$$

$$= \frac{1}{2} - c + \frac{2c}{x_{max} - K} \sum_{x_j \geq K} p(x_j)(x_j - K)$$