Neural Networks for extracting implied information from American options

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Motivation and proposal

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- Extract implied information from observed option prices with early-exercise features.
- Computing American-style option prices is generally more challenging than pricing European-style options.
- The optimization process to address the inverse problem requires solving the pricing model many thousands of times.
- Other complicating factors: negative interest rate and/or dividend yields (complex-shaped early-exercise regions)
- Implied volatility: inverse function approximated by an artificial neural network.
- Implied dividend: the inverse problem formulated as a calibration problem.

Outline

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1. Problem formulation

2. Artificial Neural Networks
   2.1 ANN for implied volatility
   2.2 ANN for implied dividend (and implied volatility)

3. Numerical results

4. Conclusions

Problem formulation

- For simplicity, we consider the Geometric Brownian Motion (GBM) process,

$$dS_t = (r - q)S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t,$$

  where S_t is the underlying price, σ is the volatility.

- The arbitrage-free value of an American option at t is

$$V_{am}(S_t, t) = \sup_{u \in [0,T]} \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)} H(K, S_u) | S_u],$$

  where $H(\cdot)$ is the payoff function with strike price K and expiration time is T.

Problem formulation

- An optimal exercise boundary $S_t^* \equiv S^*(t)$ splits the domain into early-exercise (stopping) Ω_s and continuation (holding) regions Ω_h $(\Omega = \Omega_s + \Omega_h)$.
- An American option price can be written in terms of European counterpart,

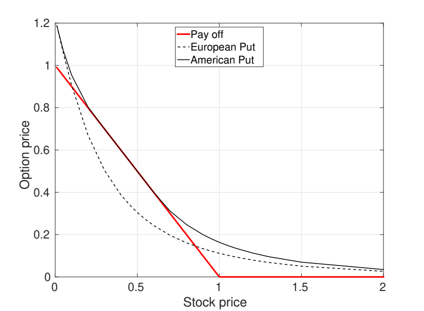
$$V_{am}^P(S_t, t) = \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)} \max(K - S_T, 0)] + \int_t^T \mathbb{E}_u^{\mathbb{Q}}[(rK - qS_u)\mathbf{1}_{\{S_u \in \Omega_s\}}]du.$$

- The second term is the *early-exercise premium*
- At the free boundary, we have,

$$V_{am}(S^*, t) = H(K, S_t^*).$$

Negative rates and dividends

- Two continuation regions may arise when both the interest rate and dividend yield become negative.
- The option value hits the payoff function twice.
- The stopping is between the two early-exercise points.



Settings: $r = -0.01$, $q = -0.06$, $\sigma = 0.2$, $T = 20$, $K = 1.0$.

Implied volatility

- The implied volatility is the level of volatility which, inserted in the pricing model, makes the market and model prices match.
- Implied volatility is an indicator for the future uncertainty of the asset prices as estimated by market participants.
- Computing the implied volatility as an inverse problem:

$$\sigma^* = BS_{am}^{-1}(V_{am}^{mkt}; S, K, \tau, r, q, \alpha),$$

  where $BS_{am}^{-1}(\cdot)$ denotes the inversion of the Black-Scholes formula, and V_{am}^{mkt} is an American option price observed in the market.
- The implied volatility inverse problem is often solved by a nonlinear root-finding method, following an iterative algorithm.
- Given an American option price observed in the market, the implied volatility σ^* is often determined by solving

$$V_{am}^{mkt} - BS_{am}(\sigma^*; S_0, K, \tau, r, q, \alpha) = 0.$$

Issues in computing implied volatility (I)

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- Existence of σ^* is guaranteed by the monotonicity of the Black-Scholes equation w.r.t to the volatility in the holding region.
- The option's Vega, becomes zero in the stopping region for American call and put options. It is well-known that,

$$|\Delta| = |\frac{\partial V_{am}}{\partial S}| = 1, \quad \text{Vega} = \frac{\partial V_{am}}{\partial \sigma} = 0.$$

- In other words, the American option prices do not depend on the volatility in the stopping regions. Consequently,

$$\frac{\partial \sigma}{\partial V_{am}} = \frac{1}{\text{Vega}} \to \infty.$$

- When inverting the American Black-Scholes pricing problem in the stopping regions, no unique solution for the implied volatility.

Issues in computing implied volatility (II)

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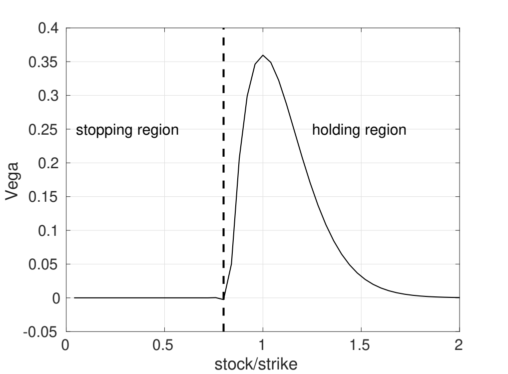
- Therefore, the definition domain should be the continuation region.



The Vega for American puts in different regions.

Implied dividend

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- Many companies pay a share of the stock value on the ex-dividend date, which causes the stock price to drop.
- This quantity is often called the actual dividend.
- The *Implied dividend* reflects how the market anticipates future dividend payments of stocks.
- The difference between actual dividends and implied dividends is similar to that between historical and implied volatility. The two parameters reflect different market aspects.
- Some companies do not pay dividends, but the corresponding options may imply a non-zero dividend, which may reflect the borrowing level of the stock.
- The borrowing costs are seen as a factor that influences the implied dividend as a function of the time or the strike price.

Issues computing implied dividend (I)

- Our approach is to estimate implied dividend and implied volatility at once.

- For European options, the implied dividend can be estimated by the put-call parity,

$$V^C_{eu}(S,t) - V^P_{eu}(S,t) = S_t e^{-q\tau} - K e^{-r\tau},$$

  so that,

$$q = -\frac{1}{\tau}\log\left(\frac{V^C_{eu} - V^P_{eu} + K e^{-r\tau}}{S_t}\right).$$

- For American-style options, the put-call parity does not hold.

- What about “de-Americanization” strategy?

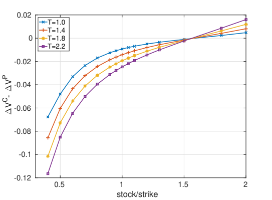
Issues computing implied dividend (II)

- Writting in terms of the early-exercise premiums, ΔV^C and ΔV^P,

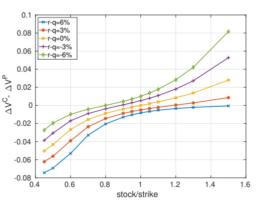
$$V^C_{am} = V^C_{eu} + \Delta V^C \text{ and } V^P_{am} = V^P_{eu} + \Delta V^P.$$

- The "deviation" from the European put-call parity is

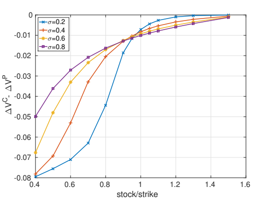
$$\text{EED} := \Delta V^C - \Delta V^P = V^C_{am} - V^P_{am} - S_t e^{-q\tau} + K e^{-r\tau}.$$



(a) EED over maturity T



(b) EED against $r - q$



(c) EED against volatility

Computing the implied dividend

- Again, model-based approach is employed: Black-Scholes.
- The dividend yield is inverted to extract the implied dividend.
- Point of departure is the American call and put model prices:

$$\begin{cases} V_{am}^{C} = BS_{am}(\sigma^{*}, q^{*}; S_0, K, \tau, r, \alpha = 1), \\ V_{am}^{P} = BS_{am}(\sigma^{*}, q^{*}; S_0, K, \tau, r, \alpha = -1), \end{cases}$$

- The system of equations is formulated as a minimization problem.
- Note that a local search based optimization method will most likely not converge when traversing those early-exercise regions.

Artificial Neural Networks (ANN)

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- ANNs are powerful function approximators.

- An ANN can be described as a composite function,

$$\mathsf{F}(\mathbf{x}|\boldsymbol{\theta}) = f^{(\ell)}(\ldots f^{(2)}(f^{(1)}(\mathbf{x};\boldsymbol{\theta}^{(1)});\boldsymbol{\theta}^{(2)});\ldots\boldsymbol{\theta}^{(\ell)}),$$

  where $\mathbf{x}$ are the input variables, $\boldsymbol{\theta}$ the hidden parameters (i.e. the weights and the biases in artificial neurons), ℓ the hidden layers, and $f^{(\ell)}(\cdot)$ the activation functions of each layer.

- Once the structure is determined, an ANN becomes a deterministic function.

- And once trained, the the evaluation of *F* is super fast.

Training an ANN

- Determining the values of the hidden parameters which will minimize a loss function.
- A popular approach for training neural networks is to employ first-order optimization algorithms.
- Gradient-based algorithms are often fast, but it may be difficult to calculate the gradients for a large test set.
- Stochastic gradient descent algorithms (SGD) randomly select a portion of the data set (saving memory).
- SGD and its variants (like Adam) are thus preferable to train the ANNs on big data sets.
- In supervised learning, the objective function is

$$\arg\min_{\boldsymbol{\theta}} L(\boldsymbol{\theta}|(\mathbf{X},\mathbf{Y})),$$

given the input-output pairs $(\mathbf{X},\mathbf{Y})$ and a user-defined loss, $L(\boldsymbol{\theta})$.

ANN for implied volatility

- Develop an ANN to approximate the inverse function in Ω_h,

$$\begin{aligned}\sigma^* &= BS_{am}^{-1}(V_{am}^{mkt}; S, K, \tau, r, q, \alpha)\\ &\approx NN(V_{am}^{mkt}; S, K, \tau, r, q, \alpha), \quad [V, S, K, \tau, r, q] \in \Omega_h.\end{aligned}$$

- The ANN must be trained based on the known market variables to approximate the unique target variable σ^*.
- The effective definition domain Ω_h corresponds to the continuation regions.
- The continuation regions are not known initially or are so complicated that there is no analytic formula to describe them.
- However, the counterpart, the early-exercise regions, can be found implicitly from the data.

Computing the holding region

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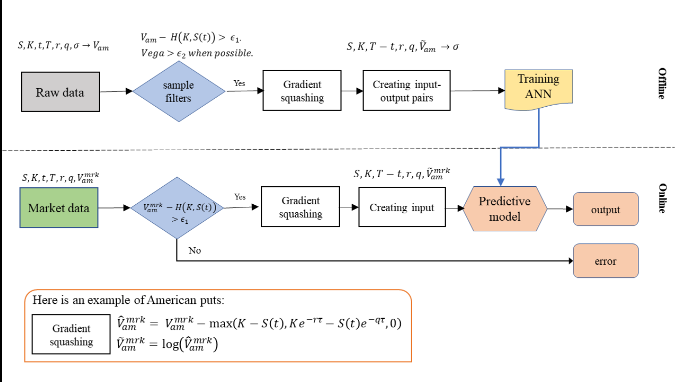
- We obtain the approximate continuation region by $\Omega_h = \Omega - \Omega_s$.
- We use two indicators to detect the samples in Ω_s:
  - The difference between the option value and the payoff.
  - The option's sensitivity Vega.
- For numerical stability, we prescribe some threshold values, ϵ_1 and ϵ_2:

$$|V_{am}(S_t, K, \tau, r, q, \sigma) - H(K, S_t)| > \epsilon_1.$$

$$\text{Vega} > \epsilon_2.$$

- The first one should cover the second one, but for robustness reason, both are enforced.
- The procedure is done off-line, prior the training phase, by filtering out the samples according to the criteria.

ANN for implied volatility



A flowchart of the ANN-based method to compute the implied volatility from American option prices.

ANN for implied information

- Determining both implied volatility and implied dividend simultaneously.
- We assume the implied volatility and the implied dividend are identical for calls and puts (given K, S_0, T, t and r):

$$\begin{cases} V_{am}^{C,mkt} - BS_{am}(\sigma^*, q^*; S_0, K, \tau, r, \alpha = 1) = 0, \\ V_{am}^{P,mkt} - BS_{am}(\sigma^*, q^*; S_0, K, \tau, r, \alpha = -1) = 0, \end{cases}$$

- Then, we have two unknown parameters to calibrate, implied volatility σ^* and implied dividend yield q^*, from a pair of American option prices, $V_{am}^{C,mkt}$ and $V_{am}^{P,mkt}$.
- The above system is reformulated as a minimization problem,

$$\arg \min_{\sigma^* \in R^+, q^* \in R} (BS_{am}(\sigma^*, q^*; \alpha = 1) - V_{am}^{C,mkt})^2 + (BS_{am}(\sigma^*, q^*; \alpha = -1) - V_{am}^{P,mkt})^2.$$

- We adapt a fast, generic and robust calibration framework, the CaNN (Calibration Neural Networks) developed in [1].

CaNN for implied information

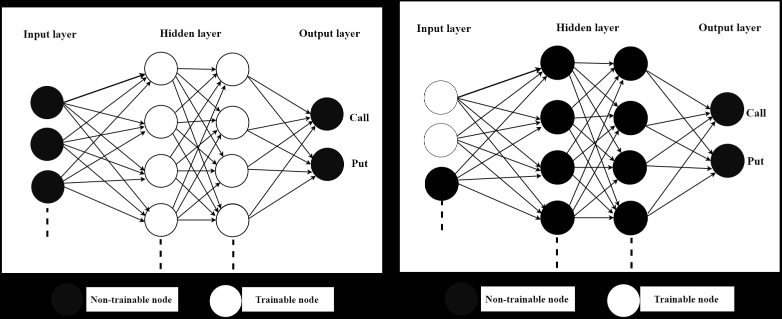
- The CaNN consists of two stages:
  - The forward pass, including the training and testing phase, approximates the American-style Black-Scholes prices.
  - The backward pass finds the two parameters (σ^*, q^*) to match the two observed American option prices, $V_{am}^{P,mkt}$ and $V_{am}^{C,mkt}$ (given K, T, S_0, r).

- We have developed one neural network providing two output values, the American call and put prices.
- The objective function is written as

$$\arg \min_{\sigma^* \in R^+, q^* \in R} (\mathrm{NN}(\sigma^*, q^*; \alpha = 1) - V_{am}^{C,mkt})^2 + (\mathrm{NN}(\sigma^*, q^*; \alpha = -1) - V_{am}^{P,mkt})^2,$$

which is used as the loss function for the backward pass.

CaNN structure



(a) Training phase (b) Calibration phase

Left: In the forward pass of the CaNN, the output layer produces two option prices. Right: In the calibration phase, the CaNN estimates the two parameters, implied volatility and implied dividend, in the original input layer.

The ANN design and training set

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- We find a balance between representation power and efficiency with the following configuration:

Hyper-parameter	Value
Hidden layers	4
Neurons (each layer)	200
Activation	Softplus
Initialization	Glorot_uniform
Optimizer	Adam
Batch size	1024

The ANN configuration.

- Other useful operations for deep NN (dropout or batch normalization) do not bring any significant benefits in our “shallow” ANN.
- Samples for the training/test set are generated by the COS method.

Settings for computing implied volatility

- Without loss of generality, we use a fixed spot price $S_0 = 1.0$.
- The two thresholds are set to $\epsilon_1 = 0.0001$ and $\epsilon_2 = 0.001$.
- We consider the measures

$$\text{MSE} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2, \quad \text{MAE} = \frac{1}{n}\sum_{i=1}^{n}|y_i - \hat{y}_i|, \quad \text{MAPE} = \frac{1}{n}\sum_{i=1}^{n}\frac{|y_i - \hat{y}_i|}{y_i}.$$

ANN	Parameters	Value range	Employed method
ANN Input	Strike, K	[0.6, 1.4]	LHS
	Time value, $\log(\hat{V}^P_{am})$	$(-11.51, -0.24)$	COS
	Time to maturity, τ	[0.05, 3.0]	LHS
	Interest rate, r	[-0.05, 0.1]	LHS
	Dividend yield, q	[-0.05, 0.1]	LHS
ANN output	Implied volatility, σ^*	$(0.01, 1.05)$	LHS

Train dataset for American options under the Black-Scholes model; The spot price $S_0 = 1$ is fixed. The upper bound of American put price is 1.2. LHS stands for Latin Hypercube Sampling.

Numerical results: implied volatility

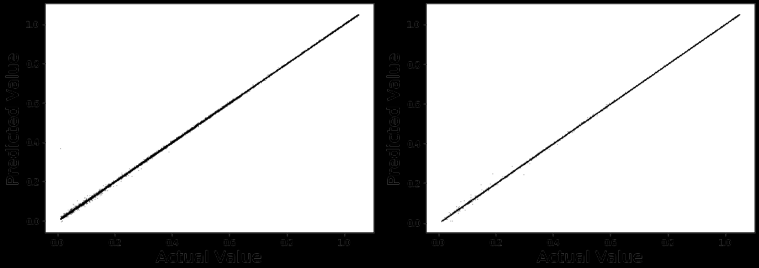
- We analyse the training and testing performance.
- The test performance is close to the train performance, suggesting that the trained ANN generalizes well for unseen data
- The ANN predicted implied volatility values approximate the true values accurately for both the train and test datasets, as is indicated by the R^2 measure.

-	MSE	MAE	MAPE	R^2
Training	$4.33 \cdot 10^{-7}$	$2.44 \cdot 10^{-4}$	$1.11 \cdot 10^{-3}$	0.999994
Testing	$4.60 \cdot 10^{-7}$	$2.51 \cdot 10^{-4}$	$1.15 \cdot 10^{-3}$	0.999993

Multiple measures are used to evaluate the performance.

Numerical results: implied volatility

- It is observed that the trained model performance tends to decrease when the pricing model parameters gets close to the upper or lower bounds
- Thus the training data set is recommended to have a wider parameter range than the test range of interest.



(a) Training (b) Testing

Left: R^2=0.999994; Right: R^2=0.999993

Settings for computing implied information

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- The loss function includes two components:

$$\text{MSE} = \frac{1}{2n}\sum_{i=1}^{n}\{(\tilde{V}^{P}_{am,i} - V^{P,mod}_{am,i})^2 + (\tilde{V}^{C}_{am,i} - V^{C,mod}_{am,i})^2\}.$$

- Training data set for the forward pass:

ANN	Parameters	Value range	Method
Forward input	Strike, K	[0.45, 1.55]	LHS
	Time to maturity, τ	[0.08, 3.05]	LHS
	Risk-free rate, r	[-0.1, 0.25]	LHS
	Dividend yield, q	[-0.1, 0.25]	LHS
	Implied volatility, σ	$(0.01, 1.05)$	LHS
Forward output	American put, V^{P}_{am}	$(0, 1.8)$	COS
	American call, V^{C}_{am}	$(0, 1.2)$	COS

We fix $S_0 = 1$, and sample strike prices K to generate different moneyness levels. The total number of the data samples is nearly one million, with 80% training, 10% validation, 10% test samples.

CaNN: performance of the forward pass

- The results, for both Calls and Puts, are highly satisfactory, achieving very good levels of precision in all the considered measures.

–	Option	MSE	MAE	MAPE	R^2
Training	Call	1.40×10^{-7}	3.00×10^{-4}	1.25×10^{-3}	0.9999965
	Put	2.54×10^{-7}	4.24×10^{-4}	1.64×10^{-3}	0.9999959
Testing	Call	1.43×10^{-7}	3.02×10^{-4}	1.27×10^{-3}	0.9999964
	Put	2.55×10^{-7}	4.26×10^{-4}	1.64×10^{-3}	0.9999959

The performance of the CaNN forward pass with two outputs.

CaNN: performance of the backward pass

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- The results suggest that the CaNN can accurately recover the implied volatility and implied dividend from “artificial market option data”.
- Even in complex scenarios (negative interest rates and/or dividend yields), CaNN recovers the true values.

K/S_0	T	r	$\sigma^\dagger$	$q^\dagger$	C_{am}^{mkt}	P_{am}^{mkt}	σ^*	q^*
1.0	0.5	-0.04	0.1	0.06	0.0146	0.0597	0.099	0.059
1.1	0.5	-0.04	0.2	-0.06	0.0255	0.1181	0.198	-0.061
1.0	0.75	0.0	0.3	-0.02	0.1119	0.0976	0.300	-0.020
1.2	1.0	-0.04	0.4	0.08	0.0603	0.3810	0.40	0.080
0.8	1.0	0.02	0.3	0.02	0.2322	0.03472	0.299	0.020
0.7	1.25	0.0	0.4	-0.04	0.3886	0.0378	0.399	-0.040

Using CaNN to extract implied volatility and implied dividend. $\dagger$ indicates the prescribed values, $*$ indicates the calibrated values.

Conclusions

- We studied a data-driven method to extract the implied volatility and/or implied dividend yield from observed market American option prices in a fast and robust way.
- For computing the American implied volatility, we propose a sophisticated ANN to approximate the inverse problem.
- The problem domain is extracted from the data, preserving the ANN offline-online decoupling advantage.
- We also propose a method for finding simultaneously implied dividend and implied volatility from American options using a calibration approach.
- The numerical experiments demonstrate that the CaNN is able to accurately extract multiple pieces of implied information from American options.
- It should be feasible to extend the approach to deal with time-dependent or discrete dividends.

References

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Acknowledgements & Questions

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Gracias!

CaNN for implied information

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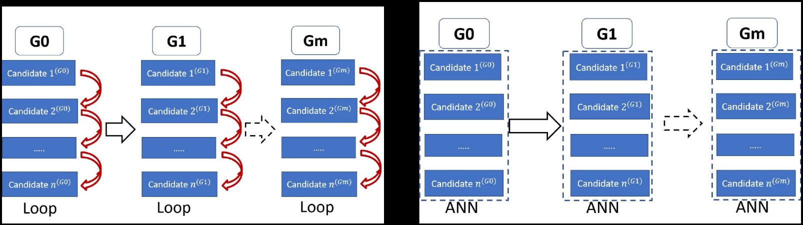
- We adapt a fast, generic and robust calibration framework, the CaNN (Calibration Neural Networks) developed in [1].
- The basic idea of the methodology is to convert the calibration of model parameters into an estimation of a neural network's hidden units.
- The model calibration and training ANNs can be reduced to solving an optimization problem.
- It enables parallel GPU computing to speed up the computations, which makes feasible to employ a global optimization technique to search the solution space.
- Here, we employ the gradient-free optimization algorithm, Differential Evolution (DE) which does not get stuck in local minima or in the stopping region.
- And DE is an inherently parallel technique.

CaNN for implied information



(a) Conventional DE (b) Parallel DE

The global optimizer DE runs in parallel within CaNN. "Gm" represents the *m*-th generation, where there are *n* candidates of to-be-optimized parameters, i.e., *n* sets of open parameters in the pricing model. These *n* sets of model parameters are independent, thus can be processed by the ANN simultaneously, instead of a for-loop, to reduce the computation time.