Deep Joint Learning valuation of Bermudan Swaptions

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Motivation and proposal

- Deep learning is increasingly used to accelerate scientific computing tasks.
- Computational finance faces demanding pricing and risk management problems.
- Derivative valuation remains one of the industry's most challenging tasks.
- Traditional numerical methods are effective but computationally expensive.
- We combine differential learning, sampled Monte Carlo labels, and joint learning.
- A novel interdependent ANN architecture is proposed for early-exercise derivatives pricing (Bermudan swaptions).

Outline

1. Problem formulation
2. Deep Joint learning for Bermudan swaptions
3. Numerical experiments
4. Conclusions

Problem formulation

- Linear Gauss Markov (LGM) model:

$$dx_t = \alpha(t)dW_t, \quad x_0 = 0,$$

with $\zeta(t) := \int_0^t \alpha^2(\tau)d\tau$.

- The *numeraire* under LGM reads

$$N(t, x_t) = \frac{1}{D(t)} \exp\left(H(t)x_t + \frac{1}{2}H^2(t)\zeta(t)\right),$$

with $D(t)$ the discount factor of time t (observed in the market)

- $H(t)$ is a curve with a similar interpretation as the mean reversion in the Hull-White model:

$$H(t) = \frac{1 - \exp(-\kappa t)}{\kappa},$$

such as κ corresponds to the Hull-White mean reversion.

LGM: Analytic pricing formulas

- Zero coupon bond at t with maturity T:

$$Z(t, x_t; T) = \frac{D(T)}{D(t)} \exp\left(-(H(T) - H(t))x_t - \frac{1}{2}(H^2(T) - H^2(t))\zeta(t)\right)$$

- Interest Rate Swap (IRS) with payment tenor $T_i, i = 1, \ldots, M$:

$$V_S(t, x_t) = \phi\left(Z(t, x_t; T) - Z(t, x_t; T_M) - K\sum_{i=1}^{M} \Delta T_i Z(t, x_t; T_i)\right)$$

- European Swaption (on the previous IRS):

$$\begin{aligned}
V_E(t, x_t) = {} & \phi Z(t, x_t, T)\mathcal{N}\left(-\phi\frac{y_T^*}{\sqrt{\zeta(T) - \zeta(t)}}\right) \\
& - \phi Z(t, x_t, T_M)\mathcal{N}\left(-\phi\frac{y_T^* + (H(T_M) - H(T))(\zeta(T) - \zeta(t))}{\sqrt{\zeta(T) - \zeta(t)}}\right) \\
& - \phi K\sum_{i=1}^{M} \Delta T_i Z(t, x_t, T_i)\mathcal{N}\left(-\phi\frac{y_T^* + (H(T_i) - H(T))(\zeta(T) - \zeta(t))}{\sqrt{\zeta(T) - \zeta(t)}}\right)
\end{aligned}$$

LGM: Bermudan swaptions

- No closed-form solution.
- Valuation of a related product, i.e., Cancellable IRS (cIRS):

$$
\begin{aligned}
V_C^p &= V_S^p - V_B^p, \\
V_C^r &= V_S^r - V_B^r,
\end{aligned}
$$

- The price of the Cancellable IRS is

$$
\begin{aligned}
\frac{V_C^p(t, x_t)}{N(t, x_t)} &= \sup_{\tau \in \{T_i / T_i > t\}} \mathbb{E}\left[\max\left(\frac{V_S^p(\tau, x_\tau)}{N(\tau, x_\tau)}, 0\right)\right], \\
\frac{V_C^r(t, x_t)}{N(t, x_t)} &= \sup_{\tau \in \{T_i / T_i > t\}} \mathbb{E}\left[\max\left(\frac{V_S^r(\tau, x_\tau)}{N(\tau, x_\tau)}, 0\right)\right].
\end{aligned}
$$

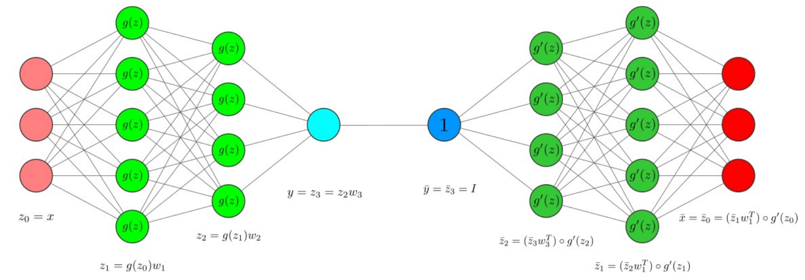
- This formulation enables the use of dynamic programming and backward induction to determine the optimal cancellation policy and, then, solve the problem.

Deep learning approach for Cancellable IRS

- We smartly combine three (individually) successful ANN components:
  - Differential Machine Learning $\rightarrow$ DANN
  - Training with *sampled* labels
  - Joint learning
- Cancellation policy: sequence of interconnected DANNs.
- Each DANN, associated with a cancellation opportunity, is used (once trained) to compute the labels of the next DANN.
- Then, the labels for the DANN of each cancellation opportunity depends on the estimations of all the “previous” DANNs.
- This specific design gets inspiration from the classical methods based on regression.
- An additional DANN approximates the final price given a newly generated samples, using joint learning feature.

Differential Machine Learning (DANN)

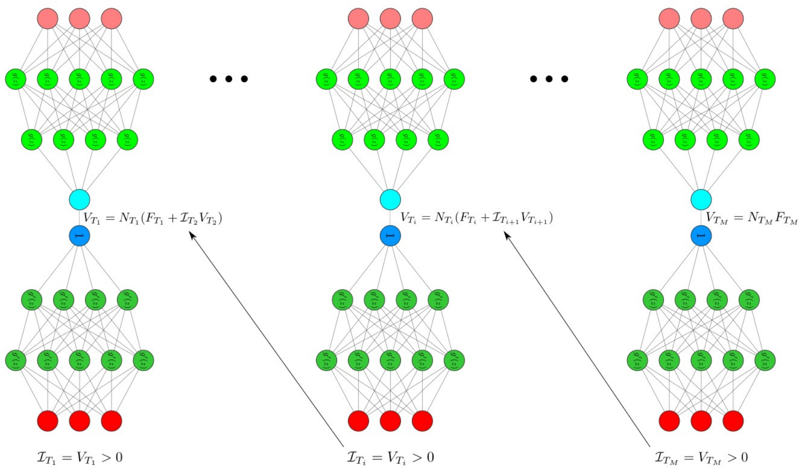
- Enlarge the network to consider (benefit from) the differentials of the output w.r.t the inputs.
- It requires the availability of those differentials.
- The loss function needs to incorporate both components.



Differential Machine Learning with ANNs.

Backward DANNs

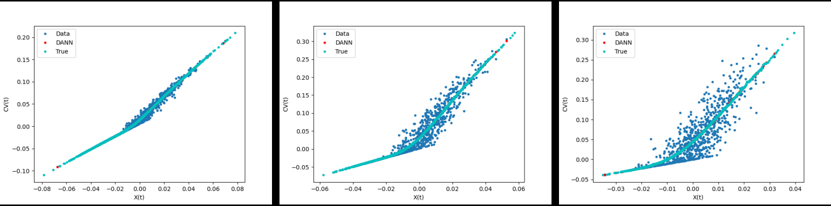
- Recursive DANN structured design.



Sequence of DANNs encapsulating the exercise policy.

Sampled (labels) payoffs

- Training the DANNs with highly noisy labels.
- The differential labels (also noisy) are obtained by AAD.



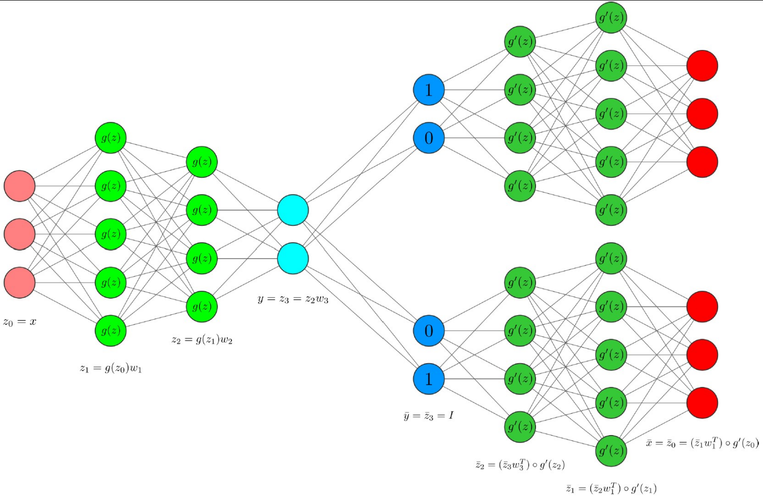
(a) $t = 7$ (b) $t = 4$ (c) $t = 1$

Sampled payoffs and Backward DANN approximation

Joint learning

- The cIRS price is estimated jointly with related financial products.
- Additional outputs include coterminal European swaptions.
- Bermudan derivatives can be viewed as combinations of European counterparts whose maturities coincide with each of the cancellation times (coterminal swaptions).
- Coterminal swaption prices are available in closed form under the LGM model.
- These exact prices provide noise-free labels, unlike sampled cIRS payoffs.
- Joint learning leverages this additional information to improve accuracy at similar computational cost.

Joint learning (with DANNs)



DANN structure considering multiple outputs, i.e., integrating the joint learning approach.

Training set generation (I)

- Training data quality depends on two key factors: the input domain and the sampling strategy.
- The input domain is typically defined according to market conditions and financial expertise.
- Sampling is critical: targeted sampling of relevant or high-error regions can outperform uniform designs under a fixed computational budget.
- Financial constraints and parameter relationships help exclude unrealistic market scenarios.
- For each generated input, a single Monte Carlo path is simulated to evolve the LGM model.

Training set generation (II)

- **Mean reversion,** $\kappa = \mathcal{U}(l_\kappa, u_\kappa)$.
- **Discount factors**: interest rate curves $\rightarrow$ discount curves

$$R(t) = \beta_0 G_0(t) + \beta_1 G_1(t) + \beta_2 G_2(t),$$

where

$$\beta_0 = \mathcal{U}(l_0, u_0), \quad \beta_1 = \mathcal{U}(l_1, u_1), \quad \beta_2 = \mathcal{U}(l_2, u_2), \quad \tau = \mathcal{U}(l_\tau, u_\tau).$$

The discount curve is constructed as $D(t) = \exp\left(-\int_0^t R(s)ds\right)$.

- **Fixed rate**. The fixed rate K, is perturbed from the ATM level:

$$K = ATM + \Delta K,$$

where

$$ATM = \frac{1 - D(T_M)}{\sum_{i=1}^{M} \Delta T_i D(T_i)},$$

with $\Delta K \in \mathcal{U}(l_K, u_K)$.

Training set generation (III)

- **LGM volatility**, α: assumed to be a piecewise constant function of the time with a dependency on κ (as observed in the market).
- How? First, Rebonato's parameterization

$$h(t) = (a + bt)\exp(-ct) + d$$

Then, we choose implied volatilities as $\Sigma_j = h\left(\frac{t_{j-1}+t_j}{2}\right)$

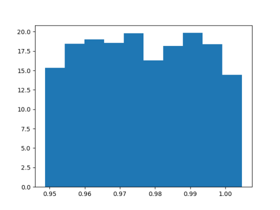
- As $\Sigma_j^2 \Delta t_j = \alpha_j^2 \int_{t_{j-1}}^{t_j} \exp\left(-2\kappa(t_j - s)\right) ds$, then $\alpha_j^2 = 2\kappa \frac{\Sigma_j^2 \Delta t_j}{1-\exp(-\kappa \Delta t_j)}$
- Finally,

$$\alpha(t) = \begin{cases} \alpha_1, & t \in (t_0, t_1], \\ \alpha_2, & t \in (t_1, t_2], \\ \vdots & \vdots \\ \alpha_J, & t \in (t_{J-1}, t_J]. \end{cases}$$

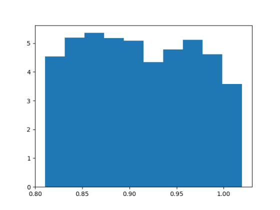
- The values of the Rebonato's model parameters are sampled by

$$a = \mathcal{U}(l_a, u_a), \quad b = \mathcal{U}(l_b, u_b), \quad c = \mathcal{U}(l_c, u_c), \quad d = \mathcal{U}(l_d, u_d).$$

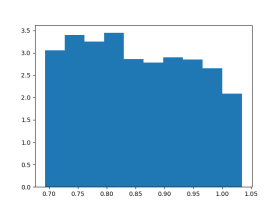
Training set generation (IV)



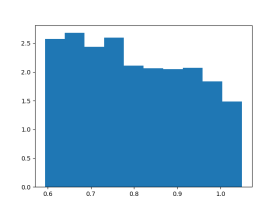
(a) $t = 1$.



(b) $t = 4$.



(c) $t = 7$.



(d) $t = 10$.

Histograms of the discount factors at different time instants.

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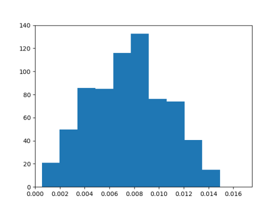
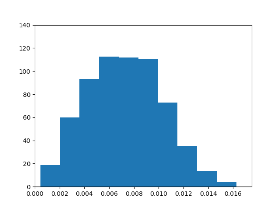
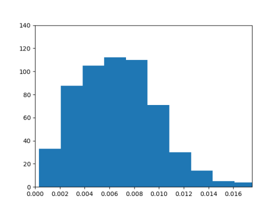
Training set generation (and V)



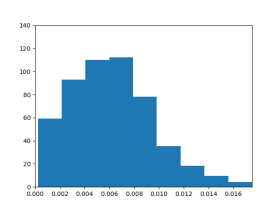
(a) α_1.



(b) α_2.



(c) α_3.



(d) α_4.

Histograms of the volatilites.

Numerical experiments

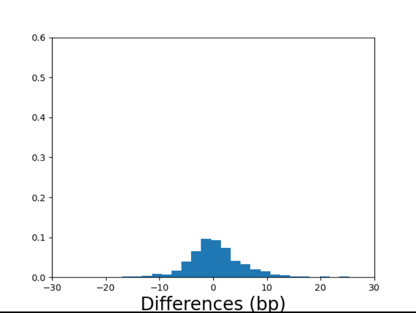
- Incremental testing in terms of the inputs' domain.
- The results are presented in the form of *differences*' histograms, including an *interquantile confidence interval*.
- The ground truth values (prices of cIRS) of the validation set are computed by a highly converged Monte Carlo pricer (acc: ~ 1 bp)
- ANNs' hyperparameters configuration:

Hyperparameter	Value
Layers	4
Neurons	32
Epochs	128
Batch size	4096

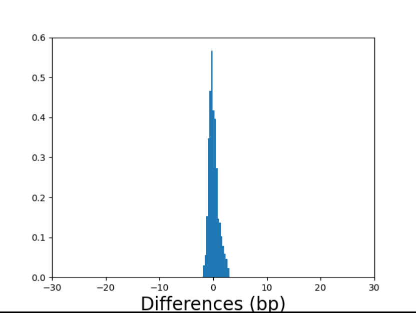
Test base cases

Test Case I	Test Case II
$l_\kappa = -0.05, \quad u_\kappa = 0.1$	$l_\kappa = -0.05, \quad u_\kappa = 0.1$
$l_a = -10^{-5}, \quad u_a = 10^{-5}$	$l_a = 10^{-5}, \quad u_a = 0.0075$
$l_b = -10^{-5}, \quad u_b = 10^{-5}$	$l_b = 0, \quad u_b = 0.0005$
$l_c = -10^{-5}, \quad u_c = 10^{-5}$	$l_c = 0, \quad u_c = 0.25$
$l_d = 0.0075 - 10^{-5}, \quad u_d = 0.0075 + 10^{-5}$	$l_d = 10^{-5}, \quad u_d = 0.0075$
$l_0 = 0.02 - 10^{-5}, \quad u_0 = 0.02 + 10^{-5}$	$l_0 = 0.02 - 10^{-5}, \quad u_0 = 0.02 + 10^{-5}$
$l_1 = -10^{-5}, \quad u_1 = 10^{-5}$	$l_1 = -10^{-5}, \quad u_1 = 10^{-5}$
$l_2 = -10^{-5}, \quad u_2 = 10^{-5}$	$l_2 = -10^{-5}, \quad u_2 = 10^{-5}$
$l_\tau = 1 - 10^{-5}, \quad u_\tau = 1 + 10^{-5}$	$l_\tau = 1 - 10^{-5}, \quad u_\tau = 1 + 10^{-5}$
$l_K = -10^{-5}, \quad u_K = 10^{-5}$	$l_K = -10^{-5}, \quad u_K = 10^{-5}$
Test Case III	**Test Case IV**
$l_\kappa = -0.05, \quad u_\kappa = 0.1$	$l_\kappa = -0.05, \quad u_\kappa = 0.1$
$l_a = 10^{-5}, \quad u_a = 0.0075$	$l_a = 10^{-5}, \quad u_a = 0.0075$
$l_b = 0, \quad u_b = 0.0005$	$l_b = 0, \quad u_b = 0.0005$
$l_c = 0, \quad u_c = 0.25$	$l_c = 0, \quad u_c = 0.25$
$l_d = 10^{-5}, \quad u_d = 0.0075$	$l_d = 10^{-5}, \quad u_d = 0.0075$
$l_0 = -0.005, \quad u_0 = 0.05$	$l_0 = -0.005, \quad u_0 = 0.05$
$l_1 = 0, \quad u_1 = 0.001$	$l_1 = 0, \quad u_1 = 0.001$
$l_2 = 0, \quad u_2 = 0.01$	$l_2 = 0, \quad u_2 = 0.01$
$l_\tau = 0.01, \quad u_\tau = 2$	$l_\tau = 0.01, \quad u_\tau = 2$
$l_K = -10^{-5}, \quad u_K = 10^{-5}$	$l_K = -0.01, \quad u_K = 0.01$

Impact of joint learning (I)



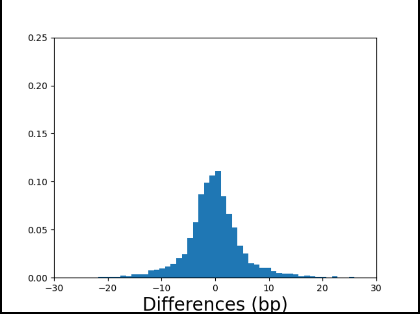
(a) $(Q_{10}Q_{90}) = (-5.7, 7.0)$.



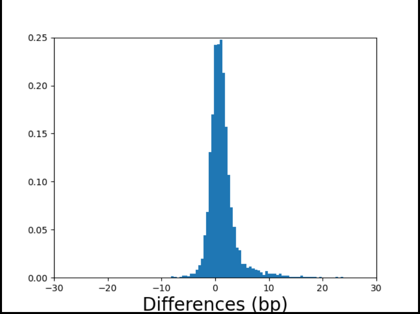
(b) $(Q_{10}, Q_{90}) = (-0.8, 1.4)$.

Pricing differences in basis points of Test Case I: Plain DANN (left) and DANN with joint learning (right).

Impact of joint learning (II)



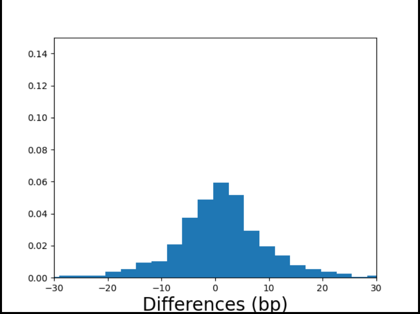
(a) $(Q_{10}, Q_{90}) = (-6.0, 5.7)$.



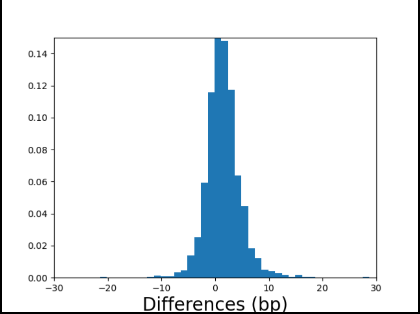
(b) $(Q_{10}, Q_{90}) = (-1.0, 3.7)$.

Pricing differences in basis points of Test Case II: Plain DANN (left) and DANN with joint learning (right).

Impact of joint learning (III)



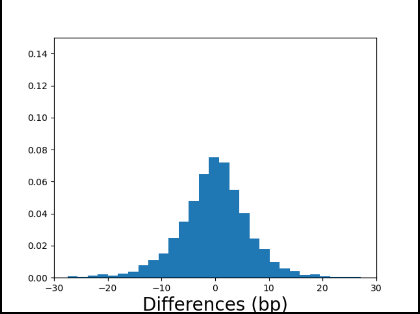
(a) $(Q_{10}, Q_{90}) = (-10.0, 12.4)$.



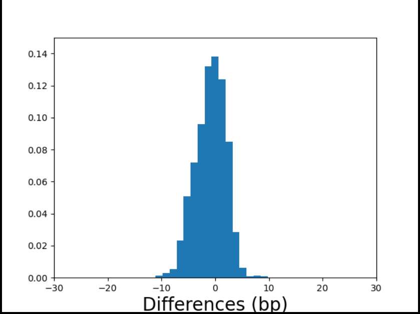
(b) $(Q_{10}, Q_{90}) = (-1.9, 5.1)$.

Pricing differences in basis points of Test Case III: Plain DANN (left) and DANN with joint learning (right).

Impact of joint learning (IV)



(a) $(Q_{10}, Q_{90}) = (-8.1, 7.4)$.



(b) $(Q_{10}, Q_{90}) = (-4.8, 2.5)$.

Pricing differences in basis points of Test Case IV: Plain DANN (left) and DANN with joint learning (right).

Impact of joint learning (IV)

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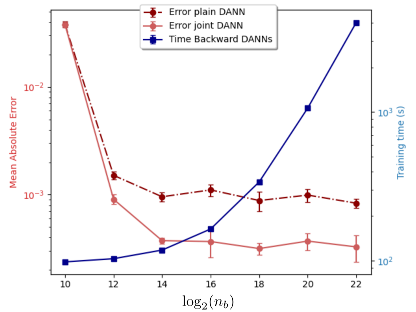
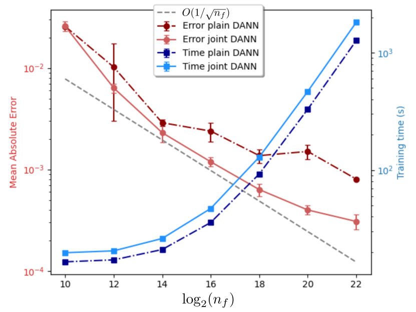
- Error distributions are centered around zero, indicating unbiased DANN predictions.
- Discount-factor and volatility inputs increase the complexity of the approximation task.
- Including the strike spread (Test Case IV) slightly improves accuracy, mainly due to far-from-ATM options, with mild skewness appearing in the error distributions.
- Joint learning consistently provides the best results, reducing both mean error and dispersion by more than 50%.

Efficiency analysis (I)

- Study of the convergence in accuracy and the computational requirements with respect to the number of samples
- Fixed $n_f = 2^{22}$ ($n_b = 2^{22}$) to isolate the precision achieved by the backward (forward) DANN.



(a) Backward DANNs.



(b) Forward DANN.

Convergence and training times in terms of the number of samples employed for training the DANNs.

Efficiency analysis (and II)

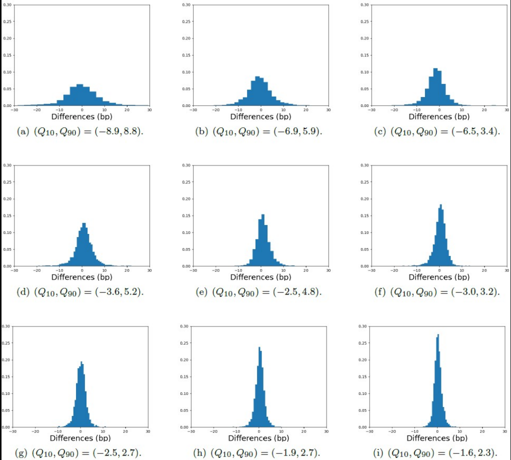
- The use of a moderate number of samples in the training of the Backward DANNs already results in good estimations, provided that the Forward DANNs is well trained.

- The joint learning strategy preserves the Monte Carlo convergence rate, $O(n_f^{-1/2})$, up to $n_f = 2^{20}$, whereas the plain DANN approach reaches saturation with significantly fewer samples.

- DANN estimators, with or without joint learning, remain robust and exhibit low variability even when trained with limited samples.

- Joint learning introduces only a marginal increase in training time while delivering a significant improvement in accuracy.

Impact of number of samples and MC paths/sample (I)



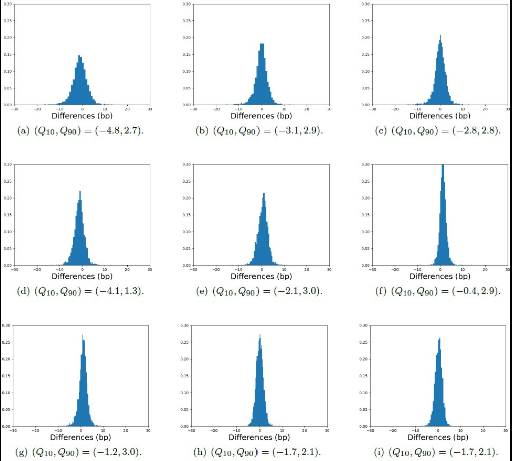
(a) $(Q_{10}, Q_{90}) = (-8.9, 8.8)$. (b) $(Q_{10}, Q_{90}) = (-6.9, 5.9)$. (c) $(Q_{10}, Q_{90}) = (-6.5, 3.4)$.

(d) $(Q_{10}, Q_{90}) = (-3.6, 5.2)$. (e) $(Q_{10}, Q_{90}) = (-2.5, 4.8)$. (f) $(Q_{10}, Q_{90}) = (-3.0, 3.2)$.

(g) $(Q_{10}, Q_{90}) = (-2.5, 2.7)$. (h) $(Q_{10}, Q_{90}) = (-1.9, 2.7)$. (i) $(Q_{10}, Q_{90}) = (-1.6, 2.3)$.

Plain DANN: Columns: $n_f = 2^{22}$ (left), $n_f = 2^{23}$ (central), $n_f = 2^{24}$ (right); Rows: $n_{MC} = 1$ (top), $n_{MC} = 4$ (middle), $n_{MC} = 16$ (bottom).

Impact of number of samples and MC paths/sample (II)



Joint DANN: Columns: $n_f = 2^{22}$ (left), $n_f = 2^{23}$ (central), $n_f = 2^{24}$ (right); Rows: $n_{MC} = 1$ (top), $n_{MC} = 4$ (middle), $n_{MC} = 16$ (bottom).

Impact of number of samples and MC paths/sample (and III)

- Error distributions are centered around zero, indicating unbiased DANN predictions.
- Discount-factor and volatility inputs increase the complexity of the approximation task.
- Including the strike spread (Test Case IV) slightly improves accuracy, mainly due to far-from-ATM options, with mild skewness appearing in the error distributions.
- Joint learning consistently provides the best results, reducing both mean error and dispersion by more than 50%.

Conclusions

- A deep learning framework for Bermudan swaption valuation is proposed.
- The approach extends standard ANNs with advanced learning enhancements.
- Key components include sampled-payoff labels and differential learning.
- A novel joint learning strategy is introduced for quantitative finance.
- Related financial products are learned simultaneously to improve accuracy at similar computational cost.
- Numerical experiments demonstrate significant gains from the joint learning approach.
- The incorporation of the industry view was critical in training set generation and joint learning.

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Acknowledgements & Questions

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